# §2 Digression: differential calculus on $\mathbb{R}^n$

**Problem 1.** **(a)** $d(x^n) = nx^{n-1}\,dx$, **(b)** $d(e^x) = e^x\,dx$, **(c)** $d(\ln x) = \frac{dx}{x}$, **(d)** $d(\sin x) = \cos x\,dx$, **(e)** $d\arctan x = \frac{dx}{1+x^2}$, **(f)** $d\sqrt{x} = \frac{dx}{2\sqrt{x}}$.

**Problem 2.** **(a)** $df = 2xdx + 2ydy + 2zdz$, **(b)** $df = 2dx - 3dy + dz$, **(c)** $df = \cos x\,dx - e^{yz}zdy - e^{yz}ydz$, **(d)** $df = -e^{-xyz}(yzdx + xzdy + xydz)$, **(e)** $df = \lambda_1 x^1 dx^1 + \ldots + \lambda_n x^n dx^n$.

**Problem 3.**
**(a)**

$$d\arctan\frac{y}{x} = \frac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = \frac{\frac{dy}{x} - y\frac{dx}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{x\,dy - y\,dx}{x^2+y^2} = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

**(b)**

$$d\ln\sqrt{x^2+y^2} = \frac{1}{2}\,d\ln(x^2+y^2) = \frac{1}{2}\,\frac{d(x^2+y^2)}{x^2+y^2} = \frac{1}{2}\,\frac{2xdx+2ydy}{x^2+y^2} = \frac{xdx+ydy}{x^2+y^2}$$

**Problem 7.**
**(a)**

$$d\boldsymbol{F} = \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dr\cos\varphi - r\sin\varphi\,d\varphi \\ dr\sin\varphi + r\cos\varphi\,d\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}\begin{pmatrix} dr \\ d\varphi \end{pmatrix}$$

**(b)** A linear transformation is invertible if and only if its matrix is invertible, hence in our case $d\boldsymbol{F}$ is invertible if the determinant

$$\begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

does not vanish. Thus $d\boldsymbol{F}$ is *not* invertible only for $r = 0$, i.e., at the point $(x, y) = (0, 0)$.
**(c)**

$$\begin{pmatrix} dr \\ d\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}^{-1}\begin{pmatrix} dx \\ dy \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\varphi & r\sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} \cos\varphi\,dx + \sin\varphi\,dy \\ -\frac{1}{r}\sin\varphi\,dx + \frac{1}{r}\cos\varphi\,dy \end{pmatrix}$$

To express $dr, d\varphi$ via $dx, dy$ is impossible when the inverse matrix does not exist, i.e. when $r = 0$.

**Problem 8**. First we have to write down this map in coordinates. In Cartesian coordinates, if $\boldsymbol{x} = (x^1, x^2, x^3)$, then

$$\boldsymbol{a} \times \boldsymbol{x} = \left( \begin{vmatrix} a^2 & a^3 \\ x^2 & x^3 \end{vmatrix}, - \begin{vmatrix} a^1 & a^3 \\ x^1 & x^3 \end{vmatrix}, \begin{vmatrix} a^1 & a^2 \\ x^1 & x^2 \end{vmatrix} \right) = \\ (a^2x^3 - a^3x^2, -a^1x^3 + a^3x^1, a^1x^2 - a^2x^1).$$

Hence $d(\boldsymbol{a} \times \boldsymbol{x}) = (a^2dx^3 - a^3dx^2, -a^1dx^3 + a^3dx^1, a^1dx^2 - a^2dx^1)$, or, in matrix notation,

$$d(\boldsymbol{a} \times \boldsymbol{x}) = \begin{pmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{pmatrix} \begin{pmatrix} dx^1 \\ dx^2 \\ dx^3 \end{pmatrix}$$

Thus the matrix of $d\boldsymbol{F}$ in these coordinates is the constant matrix

$$\begin{pmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{pmatrix}.$$

Another way to get the same answer is to notice that the product rule for the differential can be applied to the cross product, because in coordinates the cross product reduces to products of coordinates of the vectors involved. Hence we can directly apply $d$ to $\boldsymbol{a} \times \boldsymbol{x}$ getting $d(\boldsymbol{a} \times \boldsymbol{x}) = \boldsymbol{a} \times d\boldsymbol{x}$. It remains to notice that the linear operator $\boldsymbol{v} \mapsto \boldsymbol{a} \times \boldsymbol{v}$ in the Euclidean 3-dimensional space has the matrix

$$\begin{pmatrix} 0 & -a^3 & a^2 \\ a^3 & 0 & -a^1 \\ -a^2 & a^1 & 0 \end{pmatrix}$$

if $\boldsymbol{a} = (a^1, a^2, a^3)$.

**Problem 9**.

**(a)**

$$\frac{d\gamma}{dt} = (-\sin t, \cos t)$$

**(b)**

$$\frac{d\gamma}{dt} = (-2\sin 2t, 2\cos 2t, 1)$$

**(c)**

$$\frac{d\gamma}{dt} = (1, e^t)$$

**Problem 10**.

**(a)** We have

$$\dot{A}(t) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix}.$$

Hence
$$\dot{A}(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(**b**) Using the answer for $\dot{A}$ for arbitrary $t$, we obtain
$$A^{-1}\dot{A} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(The coincidence with $\dot{A}(0)$ is not by chance.)

**Problem 11**.
(**a**) Notice that we can differentiate the scalar product in $\mathbb{R}^n$ using the product rule (this follows from the formula is Cartesian coordinates: $(\boldsymbol{a}, \boldsymbol{b}) = a_1 b_1 + \ldots + a_n b_n$). Hence, since $||\boldsymbol{v}|| = 1$ is equivalent to $(\boldsymbol{v}, \boldsymbol{v}) = 1$, by differentiating we get $\frac{d}{dt}(\boldsymbol{v}, \boldsymbol{v}) = 0$, or
$$(\dot{\boldsymbol{v}}, \boldsymbol{v}) + (\boldsymbol{v}, \dot{\boldsymbol{v}}) = 0$$
i.e., $(\dot{\boldsymbol{v}}, \boldsymbol{v}) = 0$ (because the scalar product is symmetric). Thus the vectors $\boldsymbol{v}$ and $\dot{\boldsymbol{v}}$ are orthogonal.
(**b**) For the curve $\boldsymbol{v} = (\cos t, \sin t, 0)$ in $\mathbb{R}^3$ we have $(\boldsymbol{v}, \boldsymbol{v}) = \cos^2 t + \sin^2 t + 0 = 1$, so it is indeed on a unit sphere. Now, $\dot{\boldsymbol{v}} = (-\sin t, \cos t, 0)$, and $(\dot{\boldsymbol{v}}, \boldsymbol{v}) = -\sin t \cos t + \cos t \sin t = 0$. i.e, $\dot{\boldsymbol{v}} \perp \boldsymbol{v}$.

**Problem 12**. If $\boldsymbol{x} = (x, y)$, where $x, y$ are Cartesian coordinates, then
$$\begin{aligned} \boldsymbol{e}_r &= \frac{\partial \boldsymbol{x}}{\partial r} = & (\cos\varphi, \sin\varphi) &= \cos\varphi\, \boldsymbol{e}_1 + \sin\varphi\, \boldsymbol{e}_2 \\ \boldsymbol{e}_\varphi &= \frac{\partial \boldsymbol{x}}{\partial \varphi} = & (-r\sin\varphi, r\cos\varphi) &= -r\sin\varphi\, \boldsymbol{e}_1 + r\cos\varphi\, \boldsymbol{e}_2. \end{aligned}$$

Conversely, by solving linear equations we get
$$\begin{aligned} \boldsymbol{e}_1 &= \cos\varphi\, \boldsymbol{e}_r - \frac{1}{r} \sin\varphi\, \boldsymbol{e}_\varphi \\ \boldsymbol{e}_2 &= \sin\varphi\, \boldsymbol{e}_r + \frac{1}{r} \cos\varphi\, \boldsymbol{e}_\varphi. \end{aligned}$$

**Problem 14**. For the parametrized curve $r = t, \varphi = t$ we have:
(**a**)
$$\dot{\boldsymbol{x}} = \dot{r}\boldsymbol{e}_r + \dot{\varphi}\boldsymbol{e}_\varphi = \boldsymbol{e}_r + \boldsymbol{e}_\varphi,$$
and
(**b**)
$$\dot{\boldsymbol{x}} = \boldsymbol{e}_r + \boldsymbol{e}_\varphi = (\cos\varphi\, \boldsymbol{e}_1 + \sin\varphi\, \boldsymbol{e}_2) + (-r\sin\varphi\, \boldsymbol{e}_1 + r\cos\varphi\, \boldsymbol{e}_2) = \\ (\cos\varphi - r\sin\varphi)\, \boldsymbol{e}_1 + (\sin\varphi + r\cos\varphi)\, \boldsymbol{e}_2.$$

**Problem 15**. Denoting $\boldsymbol{x} = (x, y, z)$, we have:

$$\begin{aligned}
\boldsymbol{e}_r = \frac{\partial \boldsymbol{x}}{\partial r} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
&= \sin\theta\cos\varphi\,\boldsymbol{e}_1 + \sin\theta\sin\varphi\,\boldsymbol{e}_2 + \cos\theta\,\boldsymbol{e}_3 \\
\boldsymbol{e}_\theta = \frac{\partial \boldsymbol{x}}{\partial \theta} &= (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta) \\
&= r\cos\theta\cos\varphi\,\boldsymbol{e}_1 + r\cos\theta\sin\varphi\,\boldsymbol{e}_2 - r\sin\theta\,\boldsymbol{e}_3 \\
\boldsymbol{e}_\varphi = \frac{\partial \boldsymbol{x}}{\partial \varphi} &= (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0) \\
&= -r\sin\theta\sin\varphi\,\boldsymbol{e}_1 + r\sin\theta\cos\varphi\,\boldsymbol{e}_2
\end{aligned}$$

Hence

$$\begin{pmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ r\cos\theta\cos\varphi & r\cos\theta\sin\varphi & -r\sin\theta \\ -r\sin\theta\sin\varphi & r\sin\theta\cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{pmatrix}$$