§1 Differential forms in $\mathbb{R}^n$, $n \leqslant 3$.

Problem 2.

$$dz \wedge dx \wedge dy = -dx \wedge dz \wedge dy = +dx \wedge dy \wedge dz\,.$$

Similarly in other cases. (Notice that *cyclic* permutations of differentials in $dx \wedge dy \wedge dz$ do not change sign.)

Problem 5. $dx\wedge dy = (du + f'(v)dv)\wedge dv = du\wedge dv + f'(v)\, dv\wedge dv = du\wedge dv$

Problem 6. The pull-back equals $F^*\omega = F^*\big((xz + yz)\, dz\big) = (uuv + vuv)\,(du\, v + u\, dv) = u^2v^2\,(du + dv)$.

Problem 7. **(a)** Directly: $\omega = xdx + ydy = r\cos\varphi(\cos\varphi\, dr - r\sin\varphi\, d\varphi) + r\sin\varphi(\sin\varphi\, dr + r\cos\varphi\, d\varphi) = rdr$. Or: $\omega = xdx + ydy = \frac{1}{2}\,d(x^2 + y^2) = \frac{1}{2}\,d(r^2) = rdr$.

Problem 12.

$$d\omega = \frac{1}{2}\,d(x\,dy - y\,dx) = \frac{1}{2}\,(dx \wedge dy - dy \wedge dx) = \frac{1}{2}\,2\,dx \wedge dy = dx \wedge dy.$$

Problem 13. **(a)** $d\omega = (dx + 2dy) \wedge dx - 0 + (dx - dy) \wedge dz = -2dx \wedge dy + dx \wedge dz - dy \wedge dz$,
(b) $d\omega = d\Big((x^2 + y^2)^{-1}(-y\,dx + x\,dy)\Big) = -(x^2 + y^2)^{-2}(2x\,dx + 2y\,dy) \wedge (-y\,dx + x\,dy) + (x^2 + y^2)^{-1}2dx \wedge dy = 2(x^2 + y^2)^{-2}\big(-x^2\,dx \wedge dy - y^2\,dx \wedge dy + (x^2 + y^2)\,dx \wedge dy\big) = 0$,
(c) $d\omega = d\Big(2e^{x^2+y^2}\,dx \wedge dy + (x + z)\,dx \wedge dz + 5\,dy \wedge dz\Big) = 4e^{x^2+y^2}(x\,dx + y\,dy) \wedge dx \wedge dy + (dx + dz) \wedge dx \wedge dz = 0$.

Problem 14.

$$\begin{gathered}
d\omega = d\Big((x^2 + y^2 + z^2)^{-\alpha/2}(x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy)\Big) = \\
-\alpha(x^2+y^2+z^2)^{-\frac{\alpha}{2}-1}(x\,dx+y\,dy+z\,dz)\wedge(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy) + \\
(x^2 + y^2 + z^2)^{-\frac{\alpha}{2}}(dx \wedge dy \wedge dz - dy \wedge dx \wedge dz + dz \wedge dx \wedge dy) = \\
-\alpha(x^2+y^2+z^2)^{-\frac{\alpha}{2}-1}(x^2+y^2+z^2)\,dx\wedge dy\wedge dz + 3(x^2+y^2+z^2)^{-\frac{\alpha}{2}}dx\wedge dy\wedge dz = \\
(3 - \alpha)(x^2 + y^2 + z^2)^{-\alpha}dx \wedge dy \wedge dz\,.
\end{gathered}$$

It vanishes for $\alpha = 3$.