§6 Forms and vector fields on Euclidean space

Problem 1. Using the general formula for the volume form

$$dV = \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^n$$

(where $g = \det(g_{ij})$, $g_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j)$) valid for arbitrary coordinates in $\mathbb{R}^n$, check that in spherical coordinates in $\mathbb{R}^3$

$$dV = dx \wedge dy \wedge dz = r^2 \sin\theta\, dr \wedge d\theta \wedge d\varphi.$$

Problem 2. Find the formula for the area form dS for the sphere of radius R and center at the origin in $\mathbb{R}^3$
(a) using as parameters the angles θ, φ;
(a) using the parametrization $z = \sqrt{R^2 - x^2 - y^2}$ (taking x, y as parameters), valid for the upper hemisphere.

Problem 3. For a vector field $\boldsymbol{X}$ in $\mathbb{R}^3$ find the forms $\boldsymbol{X} \cdot d\boldsymbol{r}$ and $\boldsymbol{X} \cdot d\boldsymbol{S}$ working in Cartesian coordinates x, y, z:
(a) $\boldsymbol{X} = 2\boldsymbol{e}_1 - 3\boldsymbol{e}_2 + 4\boldsymbol{e}_3$
(b) $\boldsymbol{X} = (2x + z)\boldsymbol{e}_1 + 5y\boldsymbol{e}_2 + (x - y + z)\boldsymbol{e}_3$
(c) $\boldsymbol{X} = \boldsymbol{r}$ (where $\boldsymbol{r}$ is the radius-vector).

Problem 4. Deduce the formulae for the gradient in polar coordinates in $\mathbb{R}^2$ and in spherical coordinates in $\mathbb{R}^3$.

Problem 5. Find the gradient of the function $r = |\boldsymbol{r}|$ where $\boldsymbol{r}$ is the radius-vector
(a) in Cartesian coordinates in $\mathbb{R}^n$;
(b) in polar coordinates in $\mathbb{R}^2$;
(c) in spherical coordinates in $\mathbb{R}^3$.

Problem 6. Find the divergence of the vector field $\boldsymbol{X} = f(r)\boldsymbol{r}$ in $\mathbb{R}^n$ where $\boldsymbol{r}$ is the radius-vector and $r = |\boldsymbol{r}|$.

Problem 7. Find the divergence of the vector field $\boldsymbol{X} = r^\alpha \boldsymbol{r}$ in $\mathbb{R}^n$, where $\boldsymbol{r}$ is the radius-vector and $r = |\boldsymbol{r}|$. For which value of the parameter $\alpha \in \mathbb{R}$ the divergence vanishes?

Problem 8. Calculate div and curl for the following vector fields in $\mathbb{R}^3$ (given in Cartesian coordinates):
(a) $\boldsymbol{X} = (x - y + 3z)\boldsymbol{e}_1 - (2x + z)\boldsymbol{e}_2 + (-x + y + z)\boldsymbol{e}_3$
(b) $\boldsymbol{X} = (a_{11}x + a_{12}y + a_{13}z)\boldsymbol{e}_1 + (a_{12}x + a_{22}y + a_{23}z)\boldsymbol{e}_2 + (a_{13}x + a_{23}y + a_{33}z)\boldsymbol{e}_3$ where the coefficients a_{ij} are arbitrary constants;
(c) $\boldsymbol{X} = (a_{12}y + a_{13}z)\boldsymbol{e}_1 + (-a_{12}x + a_{23}z)\boldsymbol{e}_2 + (-a_{13}x - a_{23}y)\boldsymbol{e}_3$ where the coefficients a_{ij} are arbitrary constants.

Problem 9. Let $\boldsymbol{r}$ be the radius-vector in $\mathbb{R}^3$ and $r = |\boldsymbol{r}|$.
(a) Show that $\operatorname{curl}(f(r)\boldsymbol{r}) = 0$ for any function f.
(b) Show that $\operatorname{curl}(\Omega \times \boldsymbol{r}) = 2\Omega$ for any constant vector Ω.

Problem 10. Check the formula: $\operatorname{div}(f\boldsymbol{X}) = \operatorname{grad} f \cdot \boldsymbol{X} + f \operatorname{div} \boldsymbol{X}$.

Problem 11. Using the Stokes theorem show that the integral of the 1-form

$$\omega = \frac{1}{2} r^2 \, d\varphi$$

(given in polar coordinates) over the boundary of any bounded domain in $\mathbb{R}^2$ equals the area of the domain. The area of D is defined as $\int_D dx \wedge dy$ in Cartesian coordinates.

Problem 12.
(a) Apply the Stokes theorem to calculate the integral of the 2-form

$$\omega = ax \, dy \wedge dz + by \, dz \wedge dx + cz \, dx \wedge dy$$

(where a, b, c are constants) over the sphere of radius R with center at the origin in $\mathbb{R}^3$. The orientation of the sphere is induced from that for the ball given by the coordinates x, y, z. You may want to use the formula for the volume of a ball of radius R in $\mathbb{R}^3$: $\operatorname{vol} B_R = 4\pi R^3/3$.
(b) Calculate the integral of the same form over the boundary of the cube $0 \leqslant x, y, z \leqslant 1$.

Problem 13. Let S_R be the sphere of radius R with center at the origin in $\mathbb{R}^3$.
(a) Show that the integral over S_R of the 2-form $\omega = r^\alpha (x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy)$ for any $\alpha \in \mathbb{R}$ coincides with the integral over S_R of the 2-form $\sigma = R^\alpha (x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy)$. Here $r = \sqrt{x^2 + y^2 + z^2}$.
(b) Use the result of part **(a)** and the Stokes theorem to evaluate $\oint_{S_R} \omega$ for any α. *Hint:* compare with Problem 12, part **(a)**.

Problem 14. For an arbitrary $\alpha \in \mathbb{R}$ find the flux:

$$\int_{S_R} \boldsymbol{X} \cdot d\boldsymbol{S}$$

where S_R is the sphere of radius R with center at the origin in $\mathbb{R}^3$ and $\boldsymbol{X} = r^\alpha \boldsymbol{r}$. Here $\boldsymbol{r}$ is the radius-vector, $r = |\boldsymbol{r}|$. *Hint:* you can replace $\boldsymbol{X} = r^\alpha \boldsymbol{r}$ by $\boldsymbol{X}' = R^\alpha \boldsymbol{r}$ at the surface of the sphere and then apply the Ostrogradski–Gauss theorem. What is the value of the flux for $\alpha = -3$? (For $\alpha = -3$ the field $\boldsymbol{X}$ can be interpreted as the 'Coulomb force'.)