§5 Stokes theorem

Problem 1. Prove using the Stokes theorem that the integral of a 1-form $\frac{1}{2}\,(x\,dy - y\,dx)$ over the boundary of an oriented domain D in the Euclidean plane $\mathbb{R}^2$ gives the (oriented) area of D.

Problem 2. Directly verify the statement of Problem 1 for:
(a) the square with vertices $(0,0),(2,0),(2,2),(0,2)$;
(b) the parallelogram with vertices $(0,0),(4,0),(1,2),(5,2)$;
(c) the disk with centre at the origin and radius R.

Problem 3. Calculate the integral of the 2-form

$$\omega = r^{\alpha}(x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy)$$

where $r = (x^2+y^2+z^2)^{1/2}$ over the sphere of radius R with center at $O=(0,0,0)$, if $\alpha \geqslant 0$. The orientation of the sphere is given by the outward normal. *Hint:* apply the Stokes theorem.

Problem 4. Consider the 2-form in $\mathbb{R}^3\setminus\{O\}$ (here $r=(x^2+y^2+z^2)^{1/2}$):

$$\omega = \frac{x\,dy\wedge dz - y\,dx\wedge dz + z\,dx\wedge dy}{r^3}$$

(a) Calculate the integral of ω over the sphere of radius R with center at $O=(0,0,0)$. The orientation is as in the previous problem. Notice that you cannot apply the Stokes theorem to ω directly, to reduce to an integral over the ball, because ω is not defined at the origin. *Hint:* use spherical coordinates or replace ω by another form so that the Stokes theorem will be applicable to it.
(b) Check that ω is closed. Deduce from here, using the Stokes theorem, that $\oint_C \omega$ is the same for all cycles in $\mathbb{R}^3\setminus\{O\}$ representing closed oriented surfaces of the form ∂D where D is a bounded region containing the origin.

Problem 5. Consider the 1-form

$$\omega = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

in the plane. Prove that for an arbitrary function f in $\mathbb{R}^2$

$$\lim_{\varepsilon\to 0}\oint_{C_\varepsilon} f\omega = 2\pi f(O)$$

where C_ε is the circle of radius ε with center at the origin O. (We assume that f is defined and continuous near O.) What can you say about the integral of $f\omega$ over an arbitrary cycle in $\mathbb{R}^2\setminus\{O\}$?

Problem 6. If the boundary of an oriented segment (AB) is defined as the chain $(B)-(A)$ (formal difference), and the boundary of an oriented square $(ABCD)$ is defined as the chain $(AB)+(BC)+(CD)+(DA)$, check that $\partial\partial(ABCD)=0$. Do the same for a 3-cube.