§4 Integration of forms

Problem 1. Calculate the integral $\int_\gamma \omega$ if:
(a) $\gamma\colon\ t \mapsto (t^2, t^3)$, $t \in [-2, 2]$, $\omega = 2dx + dy$;
(b) $\gamma\colon\ t \mapsto (t^2, t^3)$, $t \in [-1, 1]$, $\omega = (x + y)dx + (2x - y)dy$;
(c) $\gamma\colon\ t \mapsto (t - 1, 2t^2)$, $t \in [0, 1]$, $\omega = 2ydx + (x + y)dy$.

Problem 2. Calculate the integral $\int_\gamma A$ if $\gamma\colon\ t \mapsto (2\cos t, 3\sin t, t)$, $t \in [-\pi, \pi]$, and $A = z\,dx + dy - xy\,dz$.

Problem 3. Calculate the integral of the form $\omega = dz$ over the chain $AB + BC + CA$ made of the sides of a triangle ABC in $\mathbb{R}^3$, if $A = (1, 0, 0)$, $B = (0, 2, 0)$, $C = (0, 0, 3)$.

Problem 4. Calculate the integral over the same contour as above for the form $\omega = x\,dz$

Problem 5. Given a 1-form $A = (ax + by)\,dx + (px + qy)\,dy$ on $\mathbb{R}^2$.
(a) Calculate the integral $\oint_C A$ where the closed contour C is the boundary of the square with the vertices $K = (0, 0)$, $L = (1, 0)$, $M = (1, 1)$, $N = (0, 1)$ oriented "counterclockwise" (i.e., the direction on the sides is from K to L, from L to M, from M to N, from N to K).
(b) Find the condition for $\oint_C A = 0$ as a relation on the coefficients in A.
(c) Calculate the 2-form dA.
(d) Find the condition on a, b, p, q for dA to be zero. Check that this condition coincides with the one obtained in part **(b)**.
(e) Assuming that the condition on the coefficients obtained in parts **(b)**, **(d)** is satisfied, show that $\int_{C_1} A = \int_{C_2} A = \int_{C_3} A = \int_{C_4} A$ where C_1 is the broken line KLM, C_2 is the broken line KNM, C_3 is the straight line segment $[KM]$, C_4 is the piece of the parabola $y = x^2$ between $K = (0, 0)$ and $M = (1, 1)$. In all cases the orientation is given by the direction from K to M.

Problem 6. For given pairs of bases determine whether they define the same orientation or not:
(a) $\boldsymbol{e}_1 = (1, 1), \boldsymbol{e}_2 = (-1, 1)$ and $\boldsymbol{g}_1 = (2, -7), \boldsymbol{g}_2 = (-3, 2)$
(b) $\boldsymbol{e}_1 = (1, 1, 0), \boldsymbol{e}_2 = (0, -1, 1), \boldsymbol{e}_3 = (1, 0, 4)$ and $\boldsymbol{g}_1 = (1, 0, 0), \boldsymbol{g}_2 = (0, 1, 0), \boldsymbol{g}_3 = (0, 0, 1)$
(c) $\boldsymbol{e}_1 = (1, 0, 0), \boldsymbol{e}_2 = (0, 1, 0), \boldsymbol{e}_3 = (0, 0, 1)$ and $\boldsymbol{e}_\rho, \boldsymbol{e}_\varphi, \boldsymbol{e}_z$ (corresponding to the cylindrical coordinates).

Problem 7. Check that "to define the same orientation" is an equivalence relation for bases in $\mathbb{R}^n$.

Problem 8. For spherical coordinates in $\mathbb{R}^3$ find out which order of coordinates: r, φ, θ or r, θ, φ, gives the same orientation as x, y, z.

Problem 9. Calculate the integral of the 2-form $\omega = (x+z)\, dx \wedge dy$ over a parametrized surface $\Gamma\colon\ D \to \mathbb{R}^3$ if:
(a) $\Gamma\colon\ (u,v) \mapsto (u, v, u^2 - v^2)$, where $-1 \leqslant u, v \leqslant 1$.
(b) $\Gamma\colon\ (u,v) \mapsto (u, v, 3uv)$, where $-1 \leqslant u, v \leqslant 1$.

Problem 10. Calculate the integral of the 2-form

$$\sigma = \frac{1}{3}\,(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$$

over the sphere $S^2_R\colon\ x^2 + y^2 + z^2 = R^2$ with an orientation given by the outward normal. What would happen if we choose the inward normal instead?