§3 Forms on $\mathbb{R}^n$

Problem 1. **(a)** Expand the expression $a^i e_i$ assuming that the index i runs over $1, 2, 3, 4$. Write the answer in the matrix form (in two possible ways).
(b) Do the same task (expand and write in the matrix form) for the expression $e_i F^i_j a^j$ assuming that i and j take values $1, 2, 3$.

Problem 2. Check that if a 1-form $A = A_i\, dx^i$ is the differential of a function ($A = df$ for some f), then the following relation holds:

$$\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} = 0$$

for all i, j. *Hint:* use a property of mixed partial derivatives.

Problem 3. Let $A = A_i dx^i$ be an arbitrary 1-form on $\mathbb{R}^n$. Show that its differential has the form

$$dA = \frac{1}{2}\, (\partial_i A_j - \partial_j A_i)\, dx^i \wedge dx^j\,.$$

Here ∂_i denotes the partial derivative $\partial/\partial x^i$.

Problem 4. Let $F = \frac{1}{2}\, F_{ij} dx^i \wedge dx^j$ be an arbitrary 2-form on $\mathbb{R}^n$. Show that its differential has the form

$$dF = \frac{1}{3!}\, (\partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki})\, dx^i \wedge dx^j \wedge dx^k\,.$$

Check that the coefficients $T_{ijk} = \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki}$ are antisymmetric with respect to its indices.

Problem 5. Coordinates on $\mathbb{R}^4$ considered as the 'Minkowski space-time' are traditionally denoted x^0, x^1, x^2, x^3 where $x^0 = ct$ (t being the time, c being a constant identified with the speed of light in vacuum) and $x^1 = x$, $x^2 = y$, $x^3 = z$ are Cartesian coordinates on the Euclidean 3-space.
(a) Consider a 1-form on $\mathbb{R}^4$ written as

$$A = A_i dx^i = \varphi\, cdt - A_x dx - A_y dy - A_z dz$$

(so that $A_0 = \varphi$, $A_1 = -A_x$, $A_2 = -A_y$, $A_3 = -A_z$). The function φ is known as the 'scalar potential' and the functions A_x, A_y, A_z are the components of the 'vector potential'. Find dA (you should simplify the answer as much as possible).
(b) An arbitrary 2-form on $\mathbb{R}^4$ can be written as follows:

$$F = \frac{1}{2}\, F_{ij} dx^i \wedge dx^j = E_\alpha cdt \wedge dx^\alpha - H_1 dy \wedge dz - H_2 dz \wedge dx - H_3 dx \wedge dy$$

where $\alpha = 1, 2, 3$. Write down the matrix (F_{ij}) in terms of E_α, H_α.
(c) Suppose $F = dA$. Express it as relations between E_α, H_α and φ, A_x, A_y, A_z.
(d) Show that the equation $dF = 0$ written in terms of E_α, H_α will have the appearance of (a part of) 'Maxwell's equations'

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} + \frac{\partial H_3}{\partial z} = 0$$

and

$$\begin{aligned}
\frac{1}{c}\frac{\partial H_1}{\partial t} + \frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z} &= 0 \\
\frac{1}{c}\frac{\partial H_2}{\partial t} + \frac{\partial E_1}{\partial z} - \frac{\partial E_z}{\partial x} &= 0 \\
\frac{1}{c}\frac{\partial H_3}{\partial t} + \frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y} &= 0
\end{aligned}$$

Problem 6. On $\mathbb{R}^n$ consider the $(n-1)$-form

$$\sigma = \sum_{i=1}^{n} (-1)^{i+1} x^i \, dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n.$$

Find its exterior differential.

Problem 7. On $\mathbb{R}^n$ consider the $(n-1)$-form

$$\omega = r^\alpha \sum (-1)^{i+1} x^i \, dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n$$

where $r = \left((x^1)^2 + \ldots + (x^n)^2\right)^{1/2}$. Find its exterior differential. For which values of the parameter α does $d\omega$ vanish?