§2 Digression: differential calculus on $\mathbb{R}^n$

Problem 1. Write down the expressions for
(a) $d(x^n)$, **(b)** $d(e^x)$, **(c)** $d(\ln x)$, **(d)** $d(\sin x)$, **(e)** $d\arctan x$, **(f)** $d\sqrt{x}$.

Problem 2. Find the differentials of the following functions:
(a) $f(\boldsymbol{x}) = x^2 + y^2 + z^2$, **(b)** $f(\boldsymbol{x}) = 2x - 3y + z$, **(c)** $f(\boldsymbol{x}) = \sin x - e^{yz}$, **(d)** $f(\boldsymbol{x}) = e^{-xyz}$, **(e)** $f(\boldsymbol{x}) = \dfrac{1}{2}\Big(\lambda_1(x^1)^2 + \ldots + \lambda_n(x^n)^2\Big)$

Problem 3. Calculate
(a) $d\arctan\dfrac{y}{x}$, **(b)** $d\ln\sqrt{x^2+y^2}$.

In the following problems the *derivative along a vector* $\boldsymbol{v}$, notation: $\partial_{\boldsymbol{v}} f$ or $\partial_{\boldsymbol{v}} f(\boldsymbol{x})$, means the value of the covector $df(\boldsymbol{x})$ at the vector $\boldsymbol{v}$:

$$\partial_{\boldsymbol{v}} f = df(\boldsymbol{x})(\boldsymbol{v}) = \langle df(\boldsymbol{x}), \boldsymbol{v}\rangle\,.$$

It depends both on a point $\boldsymbol{x}$ and a vector $\boldsymbol{v}$.

Problem 4. Find $\partial_{\boldsymbol{v}} f(\boldsymbol{x})$ for the function $f = x^2 + y^2$ in $\mathbb{R}^2$ if:
(a) $\boldsymbol{x} = (1,1)$, $\boldsymbol{v} = (0,0)$, **(b)** $\boldsymbol{x} = (1,1)$, $\boldsymbol{v} = (1,0)$,
(c) $\boldsymbol{x} = (1,1)$, $\boldsymbol{v} = (0,1)$, **(d)** $\boldsymbol{x} = (1,1)$, $\boldsymbol{v} = (1,2)$,
(e) $\boldsymbol{x} = (0,0)$, $\boldsymbol{v} = (1,2)$.

Problem 5. For the function $f(\boldsymbol{x}) = x^2 - y^2$ calculate the derivative along the vector $\boldsymbol{v} = (\cos\alpha, \sin\alpha)$ at $\boldsymbol{x} = (x,y)$. Here $\alpha \in \mathbb{R}$ is a parameter. Considering $\partial_{\boldsymbol{v}} f$ as a function of α, find α such that $\partial_{\boldsymbol{v}} f$ is maximal. Find the corresponding vector $\boldsymbol{v}$ and the value of the derivative $\partial_{\boldsymbol{v}} f$.

Problem 6. Given a map $\boldsymbol{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$, where $\boldsymbol{F}(x,y) = (x^2 - y^2, 2xy)$. Find the differential $d\boldsymbol{F}$. Express the answer as $d\boldsymbol{F} = \boldsymbol{A}_1\,dx + \boldsymbol{A}_2\,dy$ (with vector coefficients $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$), and in the matrix form.

Problem 7. Given a map $\boldsymbol{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{pmatrix} r \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

(a) Find the differential of $\boldsymbol{F}$. Use the matrix notation.

(b) At which values of r, θ the linear transformation $d\boldsymbol{F}$ is not invertible? To which points (x,y) in $\mathbb{R}^2$ do they correspond?

(c) Express dr, $d\theta$ via dx, dy. Is it always possible?

Problem 8. A map of a Euclidean vector space $\mathbb{R}^3$ to itself is given by $\boldsymbol{F}\colon \boldsymbol{x} \mapsto \boldsymbol{a} \times \boldsymbol{x}$ (vector product), where $\boldsymbol{a} = (a^1, a^2, a^3)$ is a constant vector. Find the matrix of the differential of $\boldsymbol{F}$. (Hint: recall the formulae for the vector product.)

Problem 9. Find the velocity vector for the following parametrized curves: **(a)** $\gamma(t) = (\cos t, \sin t)$, **(b)** $\gamma(t) = (\cos 2t, \sin 2t, t)$, **(c)** $\gamma(t) = (t, e^t)$.

Problem 10. For a path in the space of matrices

$$A(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

(**a**) find the velocity $\dot{A}(t)$ at $t = 0$; (**b**) calculate $A^{-1}\dot{A}$ for an arbitrary t.

Problem 11. (**a**) Suppose a curve $\boldsymbol{v} = \boldsymbol{v}(t)$ in $\mathbb{R}^n$ is such that $||\boldsymbol{v}|| = 1$ (i.e., the curve remains on the surface of the unit sphere with centre at the origin). Check that $\dot{\boldsymbol{v}}$ and $\boldsymbol{v}$ are orthogonal. Hint: differentiate the equation $(\boldsymbol{v}, \boldsymbol{v}) = 1$.
(**b**) Check that the curve $\boldsymbol{v}(t) = (\cos t, \sin t, 0)$ in $\mathbb{R}^3$ remains on the sphere $x^2 + y^2 + z^2 = 1$, find the velocity vector and check directly that $\dot{\boldsymbol{v}} \perp \boldsymbol{v}$.

Problem 12. For polar coordinates in the plane, express the basis $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ associated with them via the standard basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ (associated with Cartesian coordinates x, y), and conversely.

Problem 13. Sketch the basis $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ associated with polar coordinates at the following points of the plane: $x = 1, y = 0$; $x = 2, y = 0$; $x = 0, y = 1$; $x = 0, y = 2$; $x = 1/\sqrt{2}; y = 1/\sqrt{2}$; $x = 1, y = 1$.

Problem 14. A curve is given in polar coordinates: $r = t, \theta = t$. Find the velocity vector $\dot{\boldsymbol{x}}$:
(**a**) in terms of the basis $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ (associated with polar coordinates);
(**b**) in terms of the basis $\boldsymbol{e}_1, \boldsymbol{e}_2$ (associated with Cartesian coordinates x, y).
Hint: for part (**b**) you can either express the curve in Cartesian coordinates or use the result of part (**a**) and the expression of $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ via $\boldsymbol{e}_1, \boldsymbol{e}_2$.

Problem 15. Consider spherical coordinates in $\mathbb{R}^3$:

$$\begin{aligned} x &= r \sin\theta \cos\varphi \\ y &= r \sin\theta \sin\varphi \\ z &= r \cos\theta \end{aligned}$$

Express the basis $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi$ in terms of the basis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ (corresponding to the coordinates x, y, z). Write the answer in the matrix form.

Problem 16. For polar and Cartesian coordinates in $\mathbb{R}^2$ do the following:
(a) give the expression for $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ via $\boldsymbol{e}_x, \boldsymbol{e}_y$ in the matrix notation as

$$\begin{pmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{pmatrix} = T \begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{pmatrix}$$

where T is a 2×2 matrix;
(b) conversely, express $\boldsymbol{e}_x, \boldsymbol{e}_y$ via $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ writing the answer as

$$\begin{pmatrix} \boldsymbol{e}_x \\ \boldsymbol{e}_y \end{pmatrix} = S \begin{pmatrix} \boldsymbol{e}_r \\ \boldsymbol{e}_\theta \end{pmatrix}$$

(c) express the differentials dx, dy via $dr, d\theta$ writing the answer as

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = U \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

(d) conversely, express $dr, d\theta$ via dx, dy as

$$\begin{pmatrix} dr \\ d\theta \end{pmatrix} = R \begin{pmatrix} dx \\ dy \end{pmatrix}.$$

Find the relations between the matrices T, S, U, R.

Problem 17. For the surface specified by the equation $z = x^2 + y^2$ in $\mathbb{R}^3$, find a basis of the tangent plane at the points: (**a**) $P = (0,0,0)$, (**b**) $P = (1,0,1)$, (**c**) $P = (1,1,2)$. Hint: take as coordinates on the surface u, v so that $x = u, y = v, z = u^2 + v^2$ and consider the associated basis $\boldsymbol{e}_u, \boldsymbol{e}_v$.

Problem 18. Orthogonal matrices are specified by the equation $AA^T = E$, where E is the identity matrix. Consider the set $O(n)$ of the orthogonal $n \times n$ matrices as a surface in the space of all $n \times n$ matrices. Show that the tangent space for $O(n)$ at the point E is the vector space of all $n \times n$ skew-symmetric matrices.

Problem 19. **(a)** Expand the expression $a^i e_i$ assuming that the index i runs over $1, 2, 3$. Write the answer in the matrix form (in two possible ways).
(b) Do the same task (expand and write in the matrix form) for the expression $e_i F^i_j a^j$ assuming that i and j take values $1, 2$.