§1 Differential forms in $\mathbb{R}^n, n \leqslant 3$.

Problem 1. Find the exterior products:
(a) $(dx + dy) \wedge (dx - dy)$
(b) $(du + e^u dv) \wedge dv$
(c) $(2dx - dy + dz) \wedge (dy \wedge dz - 3dx \wedge dz + dx \wedge dy)$

Problem 2. Check the following in $\mathbb{R}^3$:

$$dx \wedge dy \wedge dz = dz \wedge dx \wedge dy = dy \wedge dz \wedge dx = \\ - dy \wedge dx \wedge dz = -dx \wedge dz \wedge dy = -dz \wedge dy \wedge dx.$$

Problem 3. For arbitrary three 1-forms in $\mathbb{R}^3$, $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$, $\sigma = \sigma_1 dx + \sigma_2 dy + \sigma_3 dz$, $\tau = \tau_1 dx + \tau_2 dy + \tau_3 dz$, show that

$$\omega \wedge \sigma \wedge \tau = \begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ \tau_1 & \tau_2 & \tau_3 \end{vmatrix} dx \wedge dy \wedge dz$$

Problem 4. For a 1-form $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$ and a 2-form $\sigma = K^1 dy \wedge dz + K^2 dz \wedge dx + K^3 dx \wedge dy$ (note the notation) in $\mathbb{R}^3$ show that

$$\omega \wedge \sigma = (\omega_1 K^1 + \omega_2 K^2 + \omega_3 K^3)\, dx \wedge dy \wedge dz$$

Problem 5. Show that $dx \wedge dy = du \wedge dv$, if $x = u + f(v)$, $y = v$, for an arbitrary function f.

Problem 6. A map $F\colon \mathbb{R}^2 \to \mathbb{R}^3$ is given by the formulae $(u, v) \mapsto (x, y, z)$ where $x = u, y = v, z = uv$. Calculate the pull-back $F^*\omega$ for $\omega = (xz+yz)\, dz$.

Problem 7. Express in polar coordinates r, θ in $\mathbb{R}^2$:
(a) $\omega = xdx + ydy$
(b) $\omega = xdy - ydx$
(c) $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$

Recall that spherical coordinates in $\mathbb{R}^3$ are defined by the formulae

$$x = r \sin\theta \cos\varphi, \\ y = r \sin\theta \sin\varphi, \\ z = r \cos\theta$$

where $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi \leqslant 2\pi$.

Problem 8. Express in spherical coordinates r, θ, φ in $\mathbb{R}^3$:
(a) $\sigma = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$
(b) $\sigma = \dfrac{xdy \wedge dz - ydx \wedge dz + zdx \wedge dy}{(x^2 + y^2 + z^2)^n}$
(c) $\sigma = \dfrac{z(x\,dx + y\,dy) - (x^2 + y^2)\,dz}{x^2 + y^2 + z^2}$

Problem 9. Using the formula $dx \wedge dy \wedge dz = J\,dr \wedge d\theta \wedge d\varphi$ calculate the Jacobian $J = \frac{D(x,y,z)}{D(r,\theta,\varphi)}$ for spherical coordinates in $\mathbb{R}^3$.

Problem 10. Using the formula $dx \wedge dy \wedge dz = J\,d\rho \wedge d\varphi \wedge dz$ calculate the Jacobian $J = \frac{D(x,y,z)}{D(\rho,\varphi,z)}$ for cylindrical coordinates in $\mathbb{R}^3$. (Cylindrical coordinates ρ, φ, z in $\mathbb{R}^3$ are the polar coordinates ρ, φ in the xy-plane together with the "vertical" coordinate z.)

Problem 11. Consider in $\mathbb{R}^2$ the 1-form

$$\omega = \frac{x\,dy - y\,dx}{(x^2 + y^2)^{1/2}} = \frac{x\,dy - y\,dx}{r}\,.$$

(a) Using Cartesian coordinates x, y, show that $dr \wedge \omega = dx \wedge dy$. *Hint:* first calculate dr in Cartesian coordinates.
(b) Express the form ω in the polar coordinates r, φ and use the answer to calculate $dr \wedge \omega$; check that it equals $dx \wedge dy$ as required.

Problem 12. Calculate the exterior derivative of the form

$$\omega = \frac{1}{2}\,(x\,dy - y\,dx)$$

Problem 13. Calculate the exterior derivatives for the following forms:
(a) $\omega = 2\,dx + 3\,dy - 9\,dz$,
(b) $\omega = (x + 2y)\,dx - 7\,dy + (x - y)\,dz$,
(c) $\omega = 2e^{x^2+y^2}\,dx \wedge dy + (x + z)\,dx \wedge dz + 5\,dy \wedge dz$,
(d) $\omega = \dfrac{-y\,dx + x\,dy}{x^2 + y^2}$.

Problem 14. For a 1-form ω in $\mathbb{R}^2$:

$$\omega = \frac{x\,dy - y\,dx}{r^\alpha}$$

where $r = (x^2 + y^2)^{1/2}$ and α is a real parameter:
(a) calculate $d\omega$;
(b) find α such that $d\omega = 0$.