§6 Forms and vector fields on $\mathbb{R}^n$

In this section we shall consider $\mathbb{R}^n$ as a *Euclidean space*. Recall that this means the following.

On $\mathbb{R}^n$ the *scalar product* of vectors is defined by the formula

$$(\boldsymbol{a}, \boldsymbol{b}) = a^1 b^1 + \ldots + a^n b^n \tag{1}$$

where $\boldsymbol{a} = (a^1, \ldots, a^n)$, $\boldsymbol{b} = (b^1, \ldots, b^n)$. It follows that the standard basis vectors $\boldsymbol{e}_1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{e}_n = (0, \ldots, 0, 1)$ satisfy $(\boldsymbol{e}_i, \boldsymbol{e}_j) = 0$ if $i \neq j$ and $(\boldsymbol{e}_i, \boldsymbol{e}_i) = 1$. The *length* of a vector is defined as

$$|\boldsymbol{a}| = \sqrt{(\boldsymbol{a}, \boldsymbol{a})}$$

and the *angle* between two vectors is defined by the equality

$$(\boldsymbol{a}, \boldsymbol{b}) = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\alpha$$

from where we can find $\cos\alpha$ if $(\boldsymbol{a}, \boldsymbol{b})$, $(\boldsymbol{a}, \boldsymbol{a})$, $(\boldsymbol{b}, \boldsymbol{b})$ are known. Hence the relations for $\boldsymbol{e}_i$ mean that they all have unit length and are mutually perpendicular. Length of vectors is also called *norm* or *magnitude* and alternatively denoted $||\boldsymbol{a}||$. The scalar product is alternatively denoted $\boldsymbol{a} \cdot \boldsymbol{b}$ and hence is often referred to as the 'dot product'. We shall use both notations, $\boldsymbol{a} \cdot \boldsymbol{b}$ and $(\boldsymbol{a}, \boldsymbol{b})$.

Any basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ in $\mathbb{R}^n$, not necessarily the standard basis, in which the scalar product is expressed by the same simple formula (1) where $\boldsymbol{a} = a^i \boldsymbol{e}_i$, $\boldsymbol{b} = b^i \boldsymbol{e}_i$, is called *orthonormal*. In other words, the vectors of an orthonormal basis should be mutually perpendicular and have unit length. One can see that the transition matrix between two orthonormal bases must be orthogonal, i.e., satisfy $AA^T = E$ (where A^T is the transpose of A and E denotes the identity matrix). The determinant of any orthogonal matrix equals 1 or -1. (Indeed, $AA^T = E$ implies $(\det A)^2 = 1$, since $\det A^T = \det A$.) These are standard facts from linear algebra and we are just reminding them.

Any coordinate system obtained from the standard coordinates by a linear transformation of the form $x^i = A^i_{i'} x^{i'} + b^i$ where the matrix $A = (A^i_{i'})$ is orthogonal, is called *Cartesian*. Hence the standard coordinates on $\mathbb{R}^n$ are, by definition, Cartesian, but besides them there are many other Cartesian coordinate systems. The basis of vectors associated with any Cartesian coordinates on $\mathbb{R}^n$ will be orthonormal.

6.1 Areas and volumes

Our principal goal in this subsection is to obtain convenient formulas allowing to calculate areas and volumes (for bounded domains of $\mathbb{R}^n$ or surfaces in $\mathbb{R}^n$) using arbitrary coordinates. To this end we shall start from formulas in Cartesian coordinates and then generalize them.

Let $x^1, \dots, x^n$ be Cartesian coordinates on the Euclidean space $\mathbb{R}^n$. The *volume* of any bounded domain $D \subset \mathbb{R}^n$ is defined as

$$\operatorname{vol} D := \int_D dx^1 \wedge \dots \wedge dx^n = \int_D dx^1 \dots dx^n. \tag{2}$$

The volume is defined for any domain such that the integral makes sense. It is clear that this definition does not depend on a choice of Cartesian coordinates if the orientation is not changed. (Indeed, the Jacobian will be $\det A$ where A is an orthogonal matrix, hence it is $+1$ or -1.) Note that we have actually defined the "oriented volume" or "signed volume", which depends on orientation of D. The absolute value is the usual volume. For $n = 2$ volume will be called *area*. (For $n = 1$, "volume" is length.)

Example 6.1. Let $\Pi(\boldsymbol{b}, \boldsymbol{c})$ be a parallelogram spanned by vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ in $\mathbb{R}^2$. It is the set of all points of the form $\boldsymbol{x} = A + t\boldsymbol{b} + s\boldsymbol{c}$ where $0 \leqslant t, s \leqslant 1$. A is an arbitrary (fixed) point. The area of $\Pi(\boldsymbol{b}, \boldsymbol{c})$, clearly, does not depend on A as soon as the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are given. By a direct calculation of the integral we obtain

$$\operatorname{area} \Pi(\boldsymbol{b}, \boldsymbol{c}) = \begin{vmatrix} b^1 & b^2 \\ c^1 & c^2 \end{vmatrix} \tag{3}$$

if $\boldsymbol{b} = b^1\boldsymbol{e}_1 + b^2\boldsymbol{e}_2$, $\boldsymbol{c} = c^1\boldsymbol{e}_1 + c^2\boldsymbol{e}_2$ in Cartesian coordinates.

Obviously, this generalizes to an arbitrary n: in Cartesian coordinates the volume of a parallelepiped spanned by vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ in $\mathbb{R}^n$ is

$$\operatorname{vol} \Pi(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n) = \begin{vmatrix} a_1^1 & \dots & a_1^n \\ \dots & \dots & \dots \\ a_n^1 & \dots & a_n^n \end{vmatrix}. \tag{4}$$

Here $\boldsymbol{a}_i = a_i^j \boldsymbol{e}_j$. We are working in the orthonormal basis corresponding to the Cartesian coordinates $x^1, \dots, x^n$.

Starting from (3), (4) it is possible to give an "intrinsic" expression for this volume, entirely in terms of the lengths of the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ and angles between them, i.e., in terms of the pairwise scalar products of $\boldsymbol{a}_i$.

Definition 6.1. Consider arbitrary k vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$ in $\mathbb{R}^n$. The matrix $G = G(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k)$, where

$$G(\boldsymbol{a}_1, \dots, \boldsymbol{a}_k) = \begin{pmatrix} (\boldsymbol{a}_1, \boldsymbol{a}_1) & \dots & (\boldsymbol{a}_1, \boldsymbol{a}_k) \\ \dots & \dots & \dots \\ (\boldsymbol{a}_k, \boldsymbol{a}_1) & \dots & (\boldsymbol{a}_k, \boldsymbol{a}_k) \end{pmatrix}, \tag{5}$$

is called the *Gram matrix* of the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$. Its determinant is called the *Gram determinant* of $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k$. Notation: $g = \det G$.

Theorem 6.1. *For arbitrary n vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$ in $\mathbb{R}^n$ the Gram determinant $g(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ is the square of the volume of a parallelepiped spanned by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$:*

$$(\operatorname{vol}\Pi(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n))^2 = \begin{vmatrix} (\boldsymbol{a}_1, \boldsymbol{a}_1) & \dots & (\boldsymbol{a}_1, \boldsymbol{a}_n) \\ \dots & \dots & \dots \\ (\boldsymbol{a}_n, \boldsymbol{a}_1) & \dots & (\boldsymbol{a}_n, \boldsymbol{a}_n) \end{vmatrix}. \tag{6}$$

Proof. Use the expression for $\operatorname{vol}\Pi(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ in Cartesian coordinates: by (4), it is $\det A$ where the rows of the matrix A are the arrays of coordinates of the vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_n$, respectively. For the Gram matrix $G = G(\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)$ we have

$$G = AA^T,$$

where A^T is the transpose of A. Indeed, (ij)-th entry of G is $(\boldsymbol{a}_i, \boldsymbol{a}_j) = \sum_k {a_i}^k \, {a_j}^k$, which is the (ij)-th entry of the product AA^T. $\square$

Example 6.2. For $n = 2$ we get

$$(\operatorname{area}\Pi(\boldsymbol{b}, \boldsymbol{c}))^2 = \begin{vmatrix} (\boldsymbol{b}, \boldsymbol{b}) & (\boldsymbol{b}, \boldsymbol{c}) \\ (\boldsymbol{c}, \boldsymbol{b}) & (\boldsymbol{c}, \boldsymbol{c}) \end{vmatrix} = \begin{vmatrix} |\boldsymbol{b}|^2 & |\boldsymbol{b}|\,|\boldsymbol{c}|\cos\alpha \\ |\boldsymbol{b}|\,|\boldsymbol{c}|\cos\alpha & |\boldsymbol{c}|^2 \end{vmatrix} =$$
$$|\boldsymbol{b}|^2|\boldsymbol{c}|^2 - |\boldsymbol{b}|^2|\boldsymbol{c}|^2\cos^2\alpha = |\boldsymbol{b}|^2|\boldsymbol{c}|^2(1 - \cos^2\alpha) = |\boldsymbol{b}|^2|\boldsymbol{c}|^2\sin^2\alpha\,.$$

Here α is the angle between $\boldsymbol{b}$ and $\boldsymbol{c}$. Hence the familiar formula

$$\operatorname{area}\Pi(\boldsymbol{b}, \boldsymbol{c}) = |\boldsymbol{b}|\,|\boldsymbol{c}|\sin\alpha\,.$$

The usefulness of Theorem 6.1 is twofold.

First, it gives a coordinate-free formula for the volume of a parallelepiped (the area of a parallelogram, in dimension two). Hence, as we shall see, it gives a working formula for the volume of any body in $\mathbb{R}^n$ in arbitrary coordinates.

Second, Theorem 6.1 is also applicable to a system of k vectors in $\mathbb{R}^n$ for $k \leqslant n$ (as any such system is contained in a k-dimensional Euclidean subspace). Therefore it gives a formula for a k-volume of a k-dimensional parallelepiped in the n-space. For example, it gives a formula for the area of an arbitrary parallelogram in $\mathbb{R}^3$. As we shall see, this leads to areas ($k = 2$) or volumes ($k > 2$) of k-dimensional surfaces in $\mathbb{R}^n$.

Corollary 6.1. *In arbitrary coordinates x^i in $\mathbb{R}^n$, the volume of a domain $D \subset \mathbb{R}^n$ is*

$$\operatorname{vol} D = \int_D \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^n \tag{7}$$

where $g = \det(g_{ij})$, and $g_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j)$. Here $\boldsymbol{e}_i = \frac{\partial \boldsymbol{x}}{\partial x^i}$ are the basis vectors associated with the coordinates x^i.

Proof. Indeed, if we calculate the integral in (7) as the limit of integral sums, then according to (6) the terms in the integral sum will be the volumes of small parallelipipeds spanned by the vectors $\boldsymbol{e}_1\Delta x^1, \ldots, \boldsymbol{e}_n\Delta x^n$ attached to the points of the grid. Here $\Delta x^1, \ldots, \Delta x^n$ are the increments of coordinates giving the partition of D. The parallelipipeds approximate small pieces of D between x^i and $x^i + \Delta x^i$, and their volumes, the volumes of these pieces. Passing to the limit gives the volume of D. (An alternative argument would be to consider the change of variables from x^i to some Cartesian coordinates. Then the Jacobi matrix for it will consist of the Cartesian coordinates of the vectors $\boldsymbol{e}_i$ of the basis associated with x^i. Hence its determinant, which is the Jacobian arising in the change of variables formula, is the volume of $\Pi(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ and equals $\sqrt{g}$.) $\square$

We have started our discussion from a parallelepiped spanned by arbitrary vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ in $\mathbb{R}^n$ and obtained for it a formula in Cartesian coordinates (4). Now we can give a generalization to arbitrary coordinates.

Corollary 6.2. *If $\boldsymbol{a}_i = a_i^j \boldsymbol{e}_j$, $i = 1, \ldots, n$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is an arbitrary basis, then*

$$\operatorname{vol}\Pi(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) = \sqrt{g} \begin{vmatrix} a_1^1 & \ldots & a_1^n \\ \ldots & \ldots & \ldots \\ a_n^1 & \ldots & a_n^n \end{vmatrix}, \tag{8}$$

where

$$g = \det\big((\boldsymbol{e}_i, \boldsymbol{e}_j)\big)$$

is the Gram determinant for the basis $\boldsymbol{e}_i$.

Definition 6.2. The n-form defined in arbitrary coordinates as

$$dV := \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^n \tag{9}$$

is called the *volume form* on $\mathbb{R}^n$.

The integral of dV over a bounded domain gives the volume of the domain. In particular, in any Cartesian coordinates we have $g = 1$, and we return to the original formula for volume.

Similarly, for any k-dimensional surface $M \subset \mathbb{R}^n$ we can introduce a "k-dimensional area" (or k-dimensional volume) element dS. Suppose points of the surface can be locally parametrized by independent variables $u^1, \ldots, u^k$, so that we have $\boldsymbol{x} = \boldsymbol{x}(u^1, \ldots, u^k) \in M$. Denote the corresponding basis vectors in the tangent space $T_{\boldsymbol{x}}M$ by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$, where now $\boldsymbol{e}_i = \frac{\partial \boldsymbol{x}}{\partial u^i}$. Let $h = g(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k)$ stand for the Gram determinant of $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$.

Definition 6.3. The k-form defined on M as

$$dS = \sqrt{h} du^1 \wedge \ldots \wedge du^k, \tag{10}$$

using an arbitrary parametrization $\boldsymbol{x} = \boldsymbol{x}(u^1, \ldots, u^k) \in M$, is called the *area element* for M.

The integral of the area element **by definition** is the *area* (for $k = 2$) or *volume* ($k > 2$) of a bounded region of the surface M. For a justification of such definition notice that the integral sums for an integral of dS over a piece of M will be the sums of volumes of small "tangent parallelipipeds" (areas of parallelograms for $k = 2$) spanned by $\boldsymbol{e}_1\Delta u^1, \ldots, \boldsymbol{e}_k\Delta u^k$, i.e., parallelipipeds in the tangent spaces at the points of the grid, and it should be intuitively clear that these parallelipipeds approximate "elements of surface", i.e., small pieces of M between u^i and $u^i + \Delta u^i$.

Example 6.3. In $\mathbb{R}^2$ we have

$$dS = dx \wedge dy = r\, dr \wedge d\theta$$

where x, y are Cartesian coordinates, and r, θ are polar coordinates. Indeed, for the basis $\boldsymbol{e}_r = (\cos\theta, \sin\theta)$, $\boldsymbol{e}_\theta = (-r\sin\theta, r\cos\theta)$ associated with polar coordinates we have the Gram matrix

$$G = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

(check!), hence $g = r^2$ and $\sqrt{g} = r$.

Example 6.4. In $\mathbb{R}^3$ we have

$$dV = dx \wedge dy \wedge dz = r^2 \sin\theta\, dr \wedge d\theta \wedge d\varphi = \rho d\rho \wedge d\varphi \wedge dz\,.$$

Here x, y, z are Cartesian coordinates, r, θ, φ are spherical coordinates, and ρ, φ, z are cylindrical coordinates. Indeed, for cylindrical coordinates (where $x = \rho\cos\varphi, y = \rho\sin\varphi, z = z$) we can use the result for polars in $\mathbb{R}^2$, and for spherical coordinates we arrive at the following Gram matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

corresponding to the basis $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi$ (check!). Hence for sphericals $g = r^4\sin^2\theta$ and $\sqrt{g} = r^2\sin\theta$, as stated.

Example 6.5. Consider the surface $S^2_R \subset \mathbb{R}^3$, the sphere of radius R with center at O. It can be parametrized using the angular coordinates θ, φ, as $x = R\cos\theta\cos\varphi, y = R\cos\theta\sin\varphi, z = R\sin\theta$. Then in the tangent plane we have the basis $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\varphi$, which is a part of the basis corresponding to the spherical coordinates in the ambient space $\mathbb{R}^3$ taken at a point of the sphere. Hence, immediately,

$$G = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix}$$

will be the Gram matrix for $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\varphi$. We have $h = \det G = R^4\sin^2\theta$, therefore

$$dS = R^2\, \sin\theta d\theta\, d\varphi$$

is the area element for the sphere S^2_R.

Remark 6.1. The volume element dV and the area element dS are *not* differentials of any "V" or "S", in spite of the notation. This notation is traditional and d in it has the meaning of an "element", giving the "total" volume or area after integration.

Remark 6.2. In the above definition of dV as an n-form is implicitly present a choice of orientation on $\mathbb{R}^n$. Indeed, by the very construction, the integral of dV gives oriented volume. Hence, strictly speaking, formula (9) should be used only in coordinate systems compatible with a chosen orientation. For coordinates giving the opposite orientation, an extra minus sign should be inserted: $dV = -\sqrt{g}dx^1 \wedge \ldots \wedge dx^n$. The same is true for the element of area dS on a surface. We should assume a particular choice of orientation, and if local coordinates on the surface give the wrong orientation, then (10) should be modified by an extra minus sign.

Suppose f is a function defined on a given k-dimensional surface $M \subset \mathbb{R}^n$. It is possible to consider the integral of f "over the area" of M,

$$\int_M f\, dS := \int_D f\, \sqrt{h}\, du^1 \wedge \ldots \wedge du^k \tag{11}$$

where in the RHS the integration is over an appropriate domain of parameters $u^1, \ldots, u^k$. Integrals such as (11) have been traditionally called *integrals of the first kind*. Compared to them, integrals of k-forms on $\mathbb{R}^n$ over k-surfaces are traditionally known as *integrals of the second kind*. (The names are explained by historical development, where the idea of the area element appeared before the general theory of forms.)

The differences between the two kinds of integrals are as follows: in the integral of first kind $\int_M f dS$, a function f "lives" on a particular surface M and does not have to be defined on the whole $\mathbb{R}^n$; in the integral of the second kind $\int_M \omega$, a k-form ω on $\mathbb{R}^n$ is independent of any surfaces; an integral of the second kind (i.e., integral of a k-form) does not use any extra structure (such as Euclidean) on $\mathbb{R}^n$ or a surface, while the definition of an integral of the first kind depends on the area element derived from the Euclidean structure.

However, these differences should not be overestimated. For a fixed k-dimensional surface M, any integral of the first kind $\int_M f dS$ is an integral of the k-form $f dS$ on M, which is associated with a function f on M due to the presence of a "chosen" k-form dS. On the other hand, any integral of the second kind $\int_M \omega$, for a fixed surface M can be re-written as an integral of the first kind. Indeed, $\int_M \omega = \int_D \Gamma^* \omega$ where $\Gamma\colon\ D \to \mathbb{R}^n$ is a parametrization of M, and $\Gamma^*\omega$ is a k-form on M, hence $\Gamma^*\omega = f dS$ for a suitable function f.

6.2 Forms corresponding to a vector field

6.2.1 Vector fields

Let $\boldsymbol{u}$ be a *vector field* on $\mathbb{R}^n$, i.e., a map associating with each point $\boldsymbol{x}$ a vector $\boldsymbol{u}(\boldsymbol{x})$, which we assume to depend smoothly on the point $\boldsymbol{x}$. For vector fields we shall use capital letters like $\boldsymbol{A}, \boldsymbol{F}, \boldsymbol{X}$, $\boldsymbol{Y}$, etc., or letters like $\boldsymbol{u}$, $\boldsymbol{v}$. It is instructive to visualize a vector field using a picture where vectors emanate from the corresponding points. There is a very useful hydrodynamic interpretation in which the vector $\boldsymbol{u}(\boldsymbol{x})$ at a point $\boldsymbol{x}$ is considered as the velocity of particles of a flow of some fluid in $\mathbb{R}^n$ passing through $\boldsymbol{x}$. (The flow should be stationary, in the sense that the velocity of particles at a given point $\boldsymbol{x}$ does not depend on the time.)

Remark 6.3. The term 'field' simply means a quantity depending on a point in space. A 'vector field' means vectors depending on points. Respectively, a *scalar field* is an alternative name for a usual function on $\mathbb{R}^n$ taking values in numbers ('scalars'). Differential forms, which we consider in these lectures, are also 'fields'. For example, a 1-form is a field of covectors (at each point the value of a 1-form is a covector).

Vector fields can be defined on an open set $U \subset \mathbb{R}^n$ or at the points of a surface $M \subset \mathbb{R}^n$ instead of the whole $\mathbb{R}^n$. For simplicity, we speak about vector fields on $\mathbb{R}^n$.

6.2.2 Circulation (or work) 1-form. Gradient

With each vector field $\boldsymbol{u}$ on $\mathbb{R}^n$ one can associate two differential forms: a 1-form denoted $\boldsymbol{u} \cdot d\boldsymbol{r}$ and an $(n-1)$-form denoted $\boldsymbol{u} \cdot d\boldsymbol{S}$. The definitions will follow, and the notation will be explained.

Before giving precise definitions, let us give a rough idea. Suppose we visualize $\boldsymbol{u}$ as a flow of some fluid. If γ is a path (or a 1-chain), it is natural to look for a measure of fluid that circulates *along* γ in a unit of time. Or: suppose $\boldsymbol{u}$ represents a force acting on a material particle staying on the curve γ. We are interested in the 'work of force' $\boldsymbol{u}$ along γ. Likewise, for an $(n-1)$-dimensional surface (or an $(n-1)$-chain) in $\mathbb{R}^n$ it is natural to look for a measure of fluid that passes *across* the surface in a unit of time. The answers to these two questions will be given by integrals of the forms $\boldsymbol{u} \cdot d\boldsymbol{r}$ and $\boldsymbol{u} \cdot d\boldsymbol{S}$, respectively. Notice that both questions assume Euclidean geometry: what is, precisely, 'across' a surface or 'along' a curve, or how to 'count' particles? (It is is necessary to know unit normals and tangents, as we shall see.)

Definition 6.4. The *circulation form* or the *work form* corresponding to a vector field $\boldsymbol{u}$, notation: $\boldsymbol{u} \cdot d\boldsymbol{r}$ (or $\boldsymbol{u} \cdot d\boldsymbol{x}$), is a 1-form that on every vector

$\boldsymbol{v}$ takes the value $(\boldsymbol{u}, \boldsymbol{v})$, the scalar product of $\boldsymbol{u}$ and $\boldsymbol{v}$:

$$\langle \boldsymbol{u} \cdot d\boldsymbol{r}, \boldsymbol{v} \rangle = (\boldsymbol{u}, \boldsymbol{v}). \tag{12}$$

We immediately conclude that in Cartesian coordinates, if $\boldsymbol{u} = u^i \boldsymbol{e}_i$, then

$$\boldsymbol{u} \cdot d\boldsymbol{r} = \sum_{i=1}^{n} u^i \, dx^i \tag{13}$$

(notice that we have to use the summation symbol in (12) because both indices are upper). Indeed, suppose $\boldsymbol{u} \cdot d\boldsymbol{r} = \omega = \omega_i \, dx^i$. Then the l.h.s. of (12) is $\omega_i v^i$. Comparing with (1), we get (13). In arbitrary coordinates we in the same way obtain

$$\boldsymbol{u} \cdot d\boldsymbol{r} = g_{ij} u^i \, dx^j \tag{14}$$

(summation over i, j), where $g_{ij} = (\boldsymbol{e}_i, \boldsymbol{e}_j)$.

Example 6.6. In the plane, if a vector field $\boldsymbol{X}$ is given in Cartesian coordinates x, y as $\boldsymbol{X} = X^1 \boldsymbol{e}_1 + X^2 \boldsymbol{e}_2$, then

$$\boldsymbol{X} \cdot d\boldsymbol{r} = X^1 \, dx + X^2 \, dy.$$

If $\boldsymbol{X}$ is given in polar coordinates r, θ as $\boldsymbol{X} = X^1 \boldsymbol{e}_r + X^2 \boldsymbol{e}_\theta$ (the coefficients X^1, X^2 now have a different meaning), then

$$\boldsymbol{X} \cdot d\boldsymbol{r} = X^1 \, dr + r^2 \, X^2 \, d\theta.$$

The correspondence between vectors fields and 1-forms given by the map $\boldsymbol{u} \mapsto \boldsymbol{u} \cdot d\boldsymbol{r}$ is invertible. Any 1-form $\omega = \omega_i \, dx^i$ is the circulation form for a unique vector field $\boldsymbol{u}$. All we have to do is to solve the equation (14) for the coefficients u^i. We obtain

$$\boldsymbol{u} = g^{ij} \omega_j \, \boldsymbol{e}_i \tag{15}$$

(summation over i, j), where g^{ij} (with upper indices) are the coefficients of the inverse matrix for the Gram matrix (g_{ij}).

An important example is given by the notion of the gradient of a function.

Definition 6.5. The *gradient* of a function f, notation: $\operatorname{grad} f$, is the vector field corresponding to the 1-form df:

$$\operatorname{grad} f \cdot d\boldsymbol{r} = df. \tag{16}$$

It follows that in arbitrary coordinates

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial x^i} \, \boldsymbol{e}_j \, . \tag{17}$$

Example 6.7. In Cartesian coordinates on $\mathbb{R}^n$

$$\operatorname{grad} f = \frac{\partial f}{\partial x^1}\,\boldsymbol{e}_1 + \ldots + \frac{\partial f}{\partial x^n}\,\boldsymbol{e}_n. \tag{18}$$

Example 6.8. In polar coordinates r, θ on $\mathbb{R}^2$

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\,\boldsymbol{e}_\theta. \tag{19}$$

Example 6.9. In spherical coordinates r, θ, φ on $\mathbb{R}^3$

$$\operatorname{grad} f = \frac{\partial f}{\partial r}\,\boldsymbol{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\,\boldsymbol{e}_\theta + \frac{1}{r^2\sin^2\theta}\frac{\partial f}{\partial \varphi}\,\boldsymbol{e}_\varphi. \tag{20}$$

We see that while df has a universal form in all coordinate systems, the expression for $\operatorname{grad} f$ depends on particular coordinates.

There is an alternative notation for gradient introduced by the famous Irish mathematician Hamilton. Consider grad as a linear transformation mapping functions to vector fields. A new notation for it will be $\boldsymbol{\nabla}$ (pronounced: "nabla"[1]) We have

$$\operatorname{grad} f = \boldsymbol{\nabla} f$$

and

$$\boldsymbol{\nabla} = g^{ij}\boldsymbol{e}_j\frac{\partial}{\partial x^i}\,.$$

The point is that $\boldsymbol{\nabla}$ as a vector-valued differential operator can be also treated as a "symbolic" vector field taking values in differential operators (the "components" of $\boldsymbol{\nabla}$ being $g^{ij}\frac{\partial}{\partial x^i}$). Manipulating with $\boldsymbol{\nabla}$ in this way turns out to be very convenient. In Cartesian coordinates Hamilton's $\boldsymbol{\nabla}$ takes a particularly simple form:

$$\boldsymbol{\nabla} = \boldsymbol{e}_1\frac{\partial}{\partial x^1} + \ldots + \boldsymbol{e}_n\frac{\partial}{\partial x^n}.$$

Coming back to the circulation (or work) 1-form of a vector field $\boldsymbol{u}$, let us consider its integral over a path $\gamma\colon (0,1) \to \mathbb{R}^n$, $t \mapsto \boldsymbol{x}(t)$. We have

$$\int_\gamma \boldsymbol{u}\cdot d\boldsymbol{r} = \int_0^1 \gamma^*(\boldsymbol{u}\cdot d\boldsymbol{r}) = \int_0^1 \left\langle \boldsymbol{u}\cdot d\boldsymbol{r}, \frac{d\boldsymbol{x}}{dt}\right\rangle dt = \int_0^1 \left(\boldsymbol{u}\cdot\frac{d\boldsymbol{x}}{dt}\right) dt\,.$$

If we introduce the radius-vector $\boldsymbol{r}$ (with respect to some origin $O \in \mathbb{R}^n$), then $\boldsymbol{x} = O + \boldsymbol{r}$ and $d\boldsymbol{x}/dt = d\boldsymbol{r}/dt$, so we have

$$\int_\gamma \boldsymbol{u}\cdot d\boldsymbol{r} = \int_0^1 \left(\boldsymbol{u}\cdot\frac{d\boldsymbol{r}}{dt}\right) dt\,.$$

[1] *Nabla* is a word in Aramaic, a language akin to Hebrew, meaning a "harp" or similar musical instrument of the triangle-like shape. As a mathematical symbol, with this name, $\boldsymbol{\nabla}$ was introduced by Hamilton. Sometimes this symbol is also called "del", a modification of "delta". Better use *nabla*.

The above formulas, which do not depend on a particular path γ, explain the notation for the 1-form $\boldsymbol{u} \cdot d\boldsymbol{r}$. The integral of $\boldsymbol{u} \cdot d\boldsymbol{r}$ over γ is called the *circulation* of the vector field $\boldsymbol{u}$ along the path γ.

Without any path, the expression $d\boldsymbol{x} = d\boldsymbol{r}$, the "infinitesimal displacement of a point $\boldsymbol{x}$" can be interpreted as a "1-form taking values in vectors". By definition, set $\langle d\boldsymbol{x}, \boldsymbol{v}\rangle = \boldsymbol{v}$ for any vector $\boldsymbol{v}$. Hence $d\boldsymbol{x}$ is nothing but the identity operator on vectors! Written as a vector-valued 1-form it is $d\boldsymbol{x} = \boldsymbol{e}_i dx^i$. This expression has the same form in all coordinate systems and it can be equally seen as a "vector with coefficients in 1-forms". Now, for a given vector field $\boldsymbol{u}$, the 1-form $\boldsymbol{u} \cdot d\boldsymbol{x}$ acquires a direct meaning of the scalar product of $\boldsymbol{u}$ and $d\boldsymbol{x}$; the output is a 1-form because $d\boldsymbol{x}$ as a vector is "1-form valued".

Remark 6.4. The *length* of a (piece of a) parametrized curve $\gamma\colon\ (0,1) \to \mathbb{R}^n$ between the given values of the parameter, is defined as the integral

$$\ell(\gamma) = s(\gamma) := \int_0^1 |\dot{\boldsymbol{x}}|\, dt \tag{21}$$

(ℓ and s are alternative notations for length, equally popular). For motivation one can consider integral sums. In each such sum, the terms will be the lengths of small tangent segments $\frac{d\boldsymbol{x}}{dt}\Delta t$ approximating the pieces of the curve between the values of the parameter t and $t+\Delta t$. In is natural to think that in the limit $\Delta t \to 0$ the "correct" length of the curve will be obtained. Length of curves is also referred to as *arc length.* It is clear that the integral in (21) is independent of parametrization. If the upper limit in (21) is made variable, then we arrive at the length of γ between $t = 0$ and t. It smoothly depends on t and can be taken as a new parameter. Parametrization by arc length s is known as 'natural parametrization' (s is defined up to a constant corresponding to a choice of initial point). One immediately sees that the velocity vector for the natural parametrization has unit length:

$$\left|\frac{d\boldsymbol{x}}{ds}\right| = 1\,.$$

Indeed, for any parameter t, $ds = |\dot{\boldsymbol{x}}|dt$, hence for $t = s$, $|\dot{\boldsymbol{x}}|$ must be 1.

This remark can be applied to the circulation 1-form. Suppose we choose a natural parametrization of γ. Then on γ, we have $\gamma^*(\boldsymbol{u} \cdot d\boldsymbol{x}) = (\boldsymbol{u} \cdot \boldsymbol{\tau})ds$, where $\boldsymbol{\tau} = \frac{d\boldsymbol{x}}{ds}$, and for the circulation of $\boldsymbol{u}$ along γ we obtain

$$\int_\gamma \boldsymbol{u} \cdot d\boldsymbol{x} = \int_\gamma (\boldsymbol{u} \cdot \boldsymbol{\tau})\, ds\,,$$

where the scalar product at the r.h.s. is exactly the projection of $\boldsymbol{u}$ on the tangent line to γ (the tangential component of $\boldsymbol{u}$), since $|\boldsymbol{\tau}| = 1$.

6.2.3 Flux $(n-1)$-form

Now we shall define the flux form $\boldsymbol{F}\cdot d\boldsymbol{S}$ for a vector field $\boldsymbol{F}$. (Because we are going to integrate over surfaces, where parameters are commonly denoted by letters like u, we have picked a new letter for a our vector field.) To do so, we will have to recall the above discussion of volumes in a Euclidean spaces.

First we shall define $\boldsymbol{F}\cdot d\boldsymbol{S}$ working in some Cartesian coordinates. After that we shall see that this definition has a geometric meaning independent of a choice of coordinates, which will allow to give an expression for $\boldsymbol{F}\cdot d\boldsymbol{S}$ in an arbitrary coordinate system.

Definition 6.6. Suppose $x^1,\ldots,x^n$ are Cartesian coordinates on $\mathbb{R}^n$. The *flux form* corresponding to a vector field $\boldsymbol{F}$ on $\mathbb{R}^n$, notation: $\boldsymbol{F}\cdot d\boldsymbol{S}$, is an $(n-1)$-form defined as

$$\boldsymbol{F}\cdot d\boldsymbol{S} = F^1\,dx^2\wedge dx^3\wedge\ldots\wedge dx^n - F^2\,dx^1\wedge dx^3\wedge\ldots\wedge dx^n+\ldots+ \\ (-1)^{n-1}F^n\,dx^1\wedge dx^2\wedge\ldots\wedge dx^{n-1}, \quad (22)$$

if $\boldsymbol{F}=F^i\boldsymbol{e}_i$ and $\boldsymbol{e}_i$ is the basis associated with the coordinates x^i.

The terms in the sum in (22) are obtained from $dx^1\wedge\ldots\wedge dx^n$ by omitting one differential dx^i and replacing it by the coefficient F^i with an extra sign $(-1)^{i-1}$. We have used the expression in the r.h.s. of (22) before without a connection with vector fields, as a convenient expression for $(n-1)$-forms on an n-dimensional space. In particular, it is clear that any $(n-1)$-form can be written as the flux form for some vector field.

Recall the interpretation of k-forms as alternating multilinear functions on vectors (see Section 3). For example, for the basis forms we have

$$(dx^i\wedge dx^j)(\boldsymbol{a},\boldsymbol{b}) = \frac{1}{2}\begin{vmatrix} a^i & a^j\\ b^i & b^j\end{vmatrix}$$

if $\boldsymbol{a}=a^i\boldsymbol{e}_i$, $\boldsymbol{b}=b^i\boldsymbol{e}_i$ (for $k=2$), and

$$(dx^{i_1}\wedge\ldots\wedge dx^{i_k})(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_k) = \frac{1}{k!}\begin{vmatrix} a_1^{i_1} & \ldots & a_1^{i_k}\\ \ldots & \ldots & \ldots\\ a_k^{i_1} & \ldots & a_k^{i_k}\end{vmatrix}$$

if $\boldsymbol{a}_1=a_1^i\boldsymbol{e}_i$, ..., $\boldsymbol{a}_k=a_k^i\boldsymbol{e}_i$ (for arbitrary k).

Proposition 6.1. *At every given point in $\mathbb{R}^n$, the value of the flux form corresponding to $\boldsymbol{F}$ on arbitrary vectors $\boldsymbol{a}_1$, ..., $\boldsymbol{a}_{n-1}$ is, up to a factor, the oriented volume of the parallelepiped built on $\boldsymbol{F},\boldsymbol{a}_1\ldots,\boldsymbol{a}_{n-1}$:*

$$(\boldsymbol{F}\cdot d\boldsymbol{S})(\boldsymbol{a}_1\ldots,\boldsymbol{a}_{n-1}) = \frac{1}{(n-1)!}\operatorname{vol}\Pi(\boldsymbol{F},\boldsymbol{a}_1\ldots,\boldsymbol{a}_{n-1}). \quad (23)$$

Proof. In the given Cartesian coordinate system we have

$$(\boldsymbol{F}\cdot d\boldsymbol{S})(\boldsymbol{a}_1\dots,\boldsymbol{a}_{n-1}) = \Big(F^1\,dx^2\wedge dx^3\wedge\dots\wedge dx^n - F^2\,dx^1\wedge dx^3\wedge\dots\wedge dx^n + \dots + (-1)^{n-1}F^n\,dx^1\wedge dx^2\wedge\dots\wedge dx^{n-1}\Big)(\boldsymbol{a}_1\dots,\boldsymbol{a}_{n-1}) =$$
$$F^1\,(dx^2\wedge dx^3\wedge\dots\wedge dx^n)(\boldsymbol{a}_1\dots,\boldsymbol{a}_{n-1}) - F^2\,(dx^1\wedge dx^3\wedge\dots\wedge dx^n)(\boldsymbol{a}_1\dots,\boldsymbol{a}_{n-1}) + \dots + (-1)^{n-1}F^n\,(dx^1\wedge dx^2\wedge\dots\wedge dx^{n-1})(\boldsymbol{a}_1\dots,\boldsymbol{a}_{n-1}) =$$
$$F^1\frac{1}{(n-1)!}\begin{vmatrix} a_1^2 & a_1^3 & \dots & a_1^n\\ \dots & \dots & \dots & \dots\\ a_{n-1}^2 & a_1^3 & \dots & a_{n-1}^n\end{vmatrix} - F^2\frac{1}{(n-1)!}\begin{vmatrix} a_1^1 & a_1^3 & \dots & a_1^n\\ \dots & \dots & \dots & \dots\\ a_{n-1}^1 & a_1^3 & \dots & a_{n-1}^n\end{vmatrix} + \dots +$$
$$(-1)^{n-1}F^n\frac{1}{(n-1)!}\begin{vmatrix} a_1^1 & a_1^2 & \dots & a_1^{n-1}\\ \dots & \dots & \dots & \dots\\ a_{n-1}^1 & a_1^2 & \dots & a_{n-1}^{n-1}\end{vmatrix} = \frac{1}{(n-1)!}\begin{vmatrix} F^1 & F^2 & \dots & F^n\\ a_1^1 & a_1^2 & \dots & a_1^n\\ \dots & \dots & \dots & \dots\\ a_{n-1}^1 & a_1^2 & \dots & a_{n-1}^n\end{vmatrix}.$$

The obtained determinant of order n is exactly the oriented volume of the parallelepiped $\Pi(\boldsymbol{F},\boldsymbol{a}_1\dots,\boldsymbol{a}_{n-1})$. □

Remark 6.5. The factor $\frac{1}{(n-1)!}$ appearing in the above formulae depends on convention. Namely, its source is our definition of the action of a k-form on k vectors (see Section 3), which includes the factor of $\frac{1}{k!}$. An alternative definition, without $\frac{1}{k!}$, is possible, and it will give the same formulae as above without extra factors.

Now, the r.h.s. of (23) does not depend on any coordinate system. We can take it as a 'geometric' definition of the flux form. Recalling the expression for the volume of a parallelepiped in arbitrary coordinates (8), we arrive at the following statement.

Corollary 6.3. *In arbitrary coordinates the flux form for a vector field $\boldsymbol{F}$ is given by the formula*

$$\boldsymbol{F}\cdot d\boldsymbol{S} = F^1\sqrt{g}dx^2\wedge dx^3\wedge\dots\wedge dx^n - F^2\sqrt{g}dx^1\wedge dx^3\wedge\dots\wedge dx^n + \dots + (-1)^{n-1}F^n\sqrt{g}dx^1\wedge dx^2\wedge\dots\wedge dx^{n-1},\quad (24)$$

where $g=\det\big(\boldsymbol{e}_i\cdot\boldsymbol{e}_j\big)$.

If $g=1$ (in particular, in any Cartesian coordinates), then the flux form is given by the simple formula (22).

Similarly to Proposition 6.1 we can also obtain the following.

Proposition 6.2. *For any $(n-1)$-dimensional surface $M\subset\mathbb{R}^n$, the restriction of $\boldsymbol{F}\cdot d\boldsymbol{S}$ on M equals $(\boldsymbol{F}\cdot\boldsymbol{n})\,dS$ where $\boldsymbol{n}$ is a unit normal vector for M.*

Proof. Suppose $\Gamma\colon\ D \to \mathbb{R}^n$, $\boldsymbol{x} = \boldsymbol{x}(u^1, \dots, u^{n-1})$, is a local parametrization of the surface M, where $D \subset \mathbb{R}^{n-1}$. Then the restriction of $\boldsymbol{F} \cdot d\boldsymbol{S}$ on M equals

$$\Gamma^*(\boldsymbol{F} \cdot d\boldsymbol{S}) = \sqrt{g} \sum_{i=1}^{n} (-1)^{i-1} F^i \frac{\partial x^1}{\partial u^{\alpha_1}} \cdot \ldots \cdot \frac{\partial x^{i-1}}{\partial u^{\alpha_{i-1}}} \cdot \frac{\partial x^{i+1}}{\partial u^{\alpha_{i+1}}} \cdot \ldots \cdot \frac{\partial x^n}{\partial u^{\alpha_n}}$$
$$du^{\alpha_1} \wedge \ldots \wedge du^{\alpha_{i-1}} \wedge du^{\alpha_{i+1}} \wedge \ldots \wedge du^{\alpha_n} =$$
$$\sqrt{g} \sum_{i=1}^{n} (-1)^{i-1} F^i \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \ldots & \frac{\partial x^{i-1}}{\partial u^1} & \frac{\partial x^{i+1}}{\partial u^1} & \ldots & \frac{\partial x^n}{\partial u^1} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ \frac{\partial x^1}{\partial u^{n-1}} & \ldots & \frac{\partial x^{i-1}}{\partial u^{n-1}} & \frac{\partial x^{i+1}}{\partial u^{n-1}} & \ldots & \frac{\partial x^n}{\partial u^{n-1}} \end{vmatrix} du^1 \wedge \ldots \wedge du^{n-1} =$$
$$\sqrt{g} \begin{vmatrix} F^1 & \ldots & F^n \\ \frac{\partial x^1}{\partial u^1} & \ldots & \frac{\partial x^n}{\partial u^1} \\ \ldots & \ldots & \ldots \\ \frac{\partial x^1}{\partial u^{n-1}} & \ldots & \frac{\partial x^n}{\partial u^{n-1}} \end{vmatrix} du^1 \wedge \ldots \wedge du^{n-1} =$$
$$\operatorname{vol} \Pi(\boldsymbol{F}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1})\, du^1 \wedge \ldots \wedge du^{n-1},$$

where $\boldsymbol{e}_i = \frac{\partial \boldsymbol{x}}{\partial u^i}$, $i = 1, \dots, n-1$. It remains to notice that the n-volume of the parallelepiped $\Pi(\boldsymbol{F}, \boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1})$ is the product of the $(n-1)$-volume of the base $\Pi(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1})$ and the height $\boldsymbol{F} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the unit normal vector to the plane spanned by $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1}$. Also notice that $\operatorname{vol} \Pi(\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n-1})\, du^1 \wedge \ldots \wedge du^{n-1} = dS$. $\square$

The integral of the form $\boldsymbol{F} \cdot d\boldsymbol{S}$ over an $(n-1)$-surface $M \subset \mathbb{R}^n$ is called the *flux* of the vector field $\boldsymbol{F}$ through M. As it follows from Proposition 6.2, the flux is zero if the vector field is tangent to the surface. The input of the points of M where the field $\boldsymbol{F}$ is normal to M is maximal by absolute value (relative to the magnitude of $\boldsymbol{F}$). However, this input can be positive or negative depending on whether $\boldsymbol{F}$ and $\boldsymbol{n}$ are pointing in the same or opposite directions. This agrees with the hydrodynamic interpretation outlined above, as of the "flow across the surface S".

Notice that a choice of $\boldsymbol{n}$ specifies an orientation of M. Namely, the vectors $\boldsymbol{n}, \boldsymbol{g}_1, \dots, \boldsymbol{g}_{n-1}$ (in this order), where $\boldsymbol{g}_1, \dots, \boldsymbol{g}_{n-1}$ is a basis in the tangent plane for M, should give the 'positive' orientation of $\mathbb{R}^n$. We assume everywhere that $\mathbb{R}^n$ is oriented by the standard coordinates.

Example 6.10. In $\mathbb{R}^3$, if a piece of a surface is given in the parametric form as $\boldsymbol{x} = \boldsymbol{x}(u, v)$, then the parameters u, v define an orientation of the surface via the basis $\boldsymbol{e}_u = \frac{\partial \boldsymbol{x}}{\partial u}$, $\boldsymbol{e}_v = \frac{\partial \boldsymbol{x}}{\partial v}$ of the tangent plane. The unit normal corresponding to this orientation is given by

$$\boldsymbol{n} = \frac{\boldsymbol{e}_u \times \boldsymbol{e}_v}{|\boldsymbol{e}_u \times \boldsymbol{e}_v|}. \tag{25}$$

Indeed, the rule defining the cross product is that $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b}$ should give a positive basis, and this is equivalent to $\boldsymbol{a} \times \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{b}$ giving a positive basis.

Actually, this is valid for arbitrary dimension n if the notion of a "cross product" is suitably generalized, so that it takes as arguments $n-1$ vectors instead of 2 vectors in $\mathbb{R}^3$.

Example 6.11. Find the flux of a "constant flow" along the x-axis,

$$\boldsymbol{X} = a\,\boldsymbol{e}_1$$

across a unit square in the plane P_α passing through the y-axis with an orientation specified by a unit normal $\boldsymbol{n} = (\cos\alpha, 0, \sin\alpha)$ (so the plane is at angle α with the z-axis). Solution: by Proposition 6.2, $\boldsymbol{X}\cdot d\boldsymbol{S} = (a\cos\alpha)\,dS$; thus the flux is $a\cos\alpha$. It takes the maximal value a when $\alpha = 0$, and when we rotate the plane the flux decreases to 0 for $\alpha = \pi/2$, becomes negative, and takes the value $-a$ for $\alpha = \pi$, when the orientation is "opposite to the flow".

Example 6.12. Find the flux of $\boldsymbol{u} = \boldsymbol{e}_3$ (in Cartesian coordinates) through a unit square in the plane $ax+by+cz = d$ in $\mathbb{R}^3$. Solution: the unit normal is $\boldsymbol{n} = \frac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)$; by taking its scalar product with $\boldsymbol{u} = \boldsymbol{e}_3$ we obtain $\frac{c}{\sqrt{a^2+b^2+c^2}}$. It is a constant, and integrating it over a unit square will give the same number. Answer: the flux equals $\frac{c}{\sqrt{a^2+b^2+c^2}}$.

Example 6.13. Find the flux of the vector field $\boldsymbol{F} = r^\alpha \boldsymbol{r}$ where $\boldsymbol{r}$ is the radius-vector and $r = |\boldsymbol{r}|$ through the spherical region given by $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $0 \leqslant \varphi \leqslant \frac{\pi}{4}$, for the sphere $r = R$ in $\mathbb{R}^3$. Solution: the scalar product $\boldsymbol{F}\cdot\boldsymbol{n}$ equals $r^\alpha \boldsymbol{r}\cdot\frac{\boldsymbol{r}}{|\boldsymbol{r}|} = R^{\alpha+2}R^{-1} = R^{\alpha+1}$, which is a constant on the sphere. The area the given region is one-eighth of the total area of the sphere, i.e., $\frac{1}{8}4\pi R^2$. Hence the flux is $\frac{\pi}{2}R^{\alpha+3}$.

Remark 6.6. The expression for the flux form on a surface, $(\boldsymbol{F}\cdot\boldsymbol{n})\,dS$, as

$$(\boldsymbol{F}\cdot\boldsymbol{n})\,dS = \operatorname{vol}\Pi(\boldsymbol{F}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1})\,du^1\wedge\ldots\wedge du^{n-1} = \\ \sqrt{g}\begin{vmatrix} F^1 & \ldots & F^n \\ \frac{\partial x^1}{\partial u^1} & \ldots & \frac{\partial x^n}{\partial u^1} \\ \ldots & \ldots & \ldots \\ \frac{\partial x^1}{\partial u^{n-1}} & \ldots & \frac{\partial x^n}{\partial u^{n-1}} \end{vmatrix} du^1\wedge\ldots\wedge du^{n-1} \quad (26)$$

is the most convenient for calculations if the unit normal $\boldsymbol{n}$ is not apparent geometrically as in the previous examples. The square root of the Gram determinant for the surface does not appear explicitly in (26): if we calculate $\boldsymbol{n}$ and dS separately, then in $dS = \sqrt{h}\,du^1\wedge\ldots\wedge du^{n-1}$ it will appear in the numerator and in $\boldsymbol{n}$, in the denominator; so they will be mutually cancelled.

Example 6.14. Evaluate the flux of the vector field $\boldsymbol{H} = x\boldsymbol{e}_1$ (in Cartesian coordinates) through the part M of the round paraboloid $z = 1-x^2-y^2$ above the xy-plane. Solution: consider x, y as parameters on our surface (and take the corresponding orientation); we arrive at a basis $\boldsymbol{e}_x, \boldsymbol{e}_y$ in the tangent planes, where $\boldsymbol{e}_x = \frac{\partial \boldsymbol{x}}{\partial x} = (1, 0, -2x)$, $\boldsymbol{e}_y = \frac{\partial \boldsymbol{x}}{\partial y} = (0, 1, -2y)$. Hence the "elementary flux" will be

$$\Gamma^*(\boldsymbol{H}\cdot d\boldsymbol{S}) = (\boldsymbol{H}\cdot\boldsymbol{n})\, dS = \begin{vmatrix} x & 0 & 0 \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} dx\wedge dy = x(2x)\, dx\wedge dy = 2x^2\, dx\wedge dy\,.$$

For the considered part of the paraboloid $x^2 + y^2 \leqslant 1$. Hence

$$\int_M \boldsymbol{H}\cdot d\boldsymbol{S} = \int\limits_{x^2+y^2\leqslant 1} 2x^2\, dx\wedge dy = \int_0^{2\pi}\int_0^1 2r^3\cos^2\theta dr\wedge d\theta =$$

$$\int_0^{2\pi} (\cos 2\theta + 1)\, d\theta \int_0^1 r^3\, dr = 2\pi\,\frac{1}{4} = \frac{\pi}{2}.$$

(We used polar coordinates to find the integral over the disk.) Answer: $\frac{\pi}{2}$.

6.3 Divergence, curl, and Laplacian

Shortly, divergence and curl of a vector field are particular manifestations of exterior differential. More specifically, consider a vector field $\boldsymbol{u}$ in a Euclidean space $\mathbb{R}^n$. As we have seen, there are two differential forms associated with $\boldsymbol{u}$: the flux form $\boldsymbol{u}\cdot d\boldsymbol{S}$ and the circulation form $\boldsymbol{u}\cdot d\boldsymbol{r}$. To both we can apply the operator d. Consider first the flux form.

As $\boldsymbol{u}\cdot d\boldsymbol{S}$ is an $(n-1)$-form, its differential $d\,(\boldsymbol{u}\cdot d\boldsymbol{S})$ is an n-form. Each n-form has the appearance $f\,dV$ where f is a function. As before, $dV = \sqrt{g}\,dx^1\wedge\ldots\wedge dx^n$ stands for the volume form.

Definition 6.7. The *divergence* of a vector field $\boldsymbol{u}$ is a function defined by the equality

$$d\,(\boldsymbol{u}\cdot d\boldsymbol{S}) = (\operatorname{div}\boldsymbol{u})\, dV.$$

From the above considerations (the formulas for the volume element and the flux form) it follows that in arbitrary coordinates we have the following equation for determining the function $\operatorname{div}\boldsymbol{u}$:

$$d\left(\sum (-1)^{i-1}\sqrt{g}u^i dx^1\wedge\ldots\wedge dx^{i-1}\wedge dx^{i+1}\wedge\ldots\wedge dx^n\right) = (\operatorname{div}\boldsymbol{u})\,\sqrt{g}\,dx^1\wedge\ldots\wedge dx^n\,.$$

We have calculated before the LHS; using the results of the previous calculations, we immediately get

$$\frac{\partial(\sqrt{g}u^i)}{\partial x^i}\,dx^1\wedge\ldots\wedge dx^n = (\operatorname{div}\boldsymbol{u})\,\sqrt{g}\,dx^1\wedge\ldots\wedge dx^n$$

or

$$\operatorname{div}\boldsymbol{u} = \frac{1}{\sqrt{g}}\,\frac{\partial(\sqrt{g}u^i)}{\partial x^i}\,, \tag{27}$$

which is the formula for calculating divergence in an arbitrary coordinate system.

Remark 6.7. As it is obvious from the explicit formula, divergence does not depend on the whole of the Euclidean structure (which is given, in particular coordinates, by the coefficients g_{ij}), but on the volume element only, specified by $\sqrt{g}$. There are ways of defining divergence geometrically so that this relation with volume become manifest: the divergence of a vector field $\boldsymbol{u}$, in the hydrodynamic interpretation, is the 'logarithmic rate of change of the volume form dV' under the flow of $\boldsymbol{u}$. One can make this precise.

Example 6.15. In Cartesian coordinates $x^1, \dots, x^n$ on $\mathbb{R}^n$ the general formula for divergence simplifies to

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u^i}{\partial x^i} = \frac{\partial u^1}{\partial x^1} + \dots + \frac{\partial u^n}{\partial x^n}\,. \tag{28}$$

Example 6.16. In polar coordinates r, θ on $\mathbb{R}^2$ we have

$$\operatorname{div} \boldsymbol{u} = \frac{1}{r}\left(\frac{\partial (ru^1)}{\partial r} + \frac{\partial (ru^2)}{\partial \theta}\right) = \frac{1}{r}\frac{\partial (ru^1)}{\partial r} + \frac{\partial u^2}{\partial \theta} \tag{29}$$

for $\boldsymbol{u} = u^1\boldsymbol{e}_r + u^2\boldsymbol{e}_\theta$.

Example 6.17. In spherical coordinates r, θ, φ on $\mathbb{R}^3$ we have

$$\operatorname{div} \boldsymbol{u} = \frac{1}{r^2\sin\theta}\left(\frac{\partial (r^2\sin\theta\, u^1)}{\partial r} + \frac{\partial (r^2\sin\theta\, u^2)}{\partial \theta} + \frac{\partial (r^2\sin\theta\, u^3)}{\partial \varphi}\right) = \tag{30}$$

$$\frac{1}{r^2}\frac{\partial (r^2\, u^1)}{\partial r} + \frac{1}{\sin\theta}\frac{\partial (\sin\theta\, u^2)}{\partial \theta} + \frac{\partial u^3}{\partial \varphi} \tag{31}$$

for $\boldsymbol{u} = u^1\boldsymbol{e}_r + u^2\boldsymbol{e}_\theta + u^3\boldsymbol{e}_\varphi$.

One of the applications of the formulas obtained is to the Laplace operator.

Definition 6.8. The *Laplace operator*, or the *Laplacian*, notation: Δ (not to be confused with the same notation for the increment), acts on functions as follows:

$$\Delta f = \operatorname{div} \operatorname{grad} f\,. \tag{32}$$

We immediately obtain from (17) and (27) that in arbitrary coordinates

$$\Delta f = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^j}\right)\,. \tag{33}$$

Notice that without prior knowing that Δf does not depend on a choice of coordinates (as defined by (32), where both operations, div and grad, do not depend on coordinates), it would be not easy to show it by a direct change of coordinates in (33).

Example 6.18. Laplacian in Cartesian coordinates:

$$\Delta f = \delta^{ij} \frac{\partial}{\partial x^i} \frac{\partial f}{\partial x^j} = \frac{\partial^2 f}{\partial x_1^2} + \ldots + \frac{\partial^2 f}{\partial x_n^2}$$

on $\mathbb{R}^n$, where it is customary to use lower indices for the coordinates: $x_1 = x^1$, $\ldots$, $x_n = x^n$. (Here $\delta_{ij} = \delta^{ij}$ equals 1 for $i = j$ and 0, for $i \neq j$.)

Example 6.19. In particular, in traditional notation for $\mathbb{R}^3$,

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\,.$$

Applying (33) or combining the results of Examples 6.8, 6.16, and 6.9, 6.17, we arrive at the important formulas for the Laplacian in polars on the plane and in spherical coordinates in 3-space.

Example 6.20. In polar coordinates on $\mathbb{R}^2$:

$$\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2}\,. \tag{34}$$

Example 6.21. In cylindrical coordinates on $\mathbb{R}^3$:

$$\Delta f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}\,. \tag{35}$$

Example 6.22. In spherical coordinates on $\mathbb{R}^3$:

$$\Delta f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \varphi^2}\,. \tag{36}$$

These formulas have lots of applications in mathematical physics, where the Laplace operator appear in fundamental differential equations. The possibility of looking for solutions of these equations working in the coordinate system most appropriate for a particular problem is of paramount importance.

Example 6.23. Let us find $\Delta(r^\alpha)$, where $r = |\boldsymbol{r}|$ is the radius, for $\mathbb{R}^2$ and $\mathbb{R}^3$. According to (34) and (36) we obtain:

$$\Delta(r^\alpha) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial (r^\alpha)}{\partial r} \right) = \frac{1}{r} \frac{\partial (\alpha r^\alpha)}{\partial r} = \alpha^2 r^{\alpha-2} \quad \text{for } \mathbb{R}^2$$

and

$$\Delta(r^\alpha) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial (r^\alpha)}{\partial r} \right) = \frac{1}{r^2} \frac{\partial (\alpha r^{\alpha+1})}{\partial r} = \alpha(\alpha+1) r^{\alpha-2} \quad \text{for } \mathbb{R}^3\,.$$

Hence, on $\mathbb{R}^3$, $\Delta \frac{1}{r} = 0$ for $r \neq 0$. On the other hand, on $\mathbb{R}^2$ no power of r is annihilated by the Laplace operator except for the trivial case $\alpha = 0$ (constant).

Example 6.24. Consider $f = \ln r$ on $\mathbb{R}^2$. We have

$$\Delta \ln r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial(\ln r)}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\,(1) = 0\,.$$

The Laplace operator, as well as divergence and gradient, make sense not only for the Euclidean space, but also for surfaces therein, since all what is needed is the notion of the scalar product, which surfaces inherit from the ambient space.

Example 6.25. Using the angles θ, φ as local coordinates on the sphere of radius R with center at the origin in $\mathbb{R}^3$, we have the Gram matrix

$$(g_{ij}) = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix},$$

so $\sqrt{g} = R^2\sin\theta$, and the inverse matrix g^{ij} is diagonal with the entries $R^{-2}, (R\sin\theta)^{-2}$. Hence for the gradient of a function on the sphere we have

$$\operatorname{grad} f = \frac{1}{R^2}\left(\frac{\partial f}{\partial \theta}\,\boldsymbol{e}_\theta + \frac{1}{\sin^2\theta}\frac{\partial f}{\partial \varphi}\,\boldsymbol{e}_\varphi\right).$$

For the divergence of a vector field $\boldsymbol{u} = u^1\boldsymbol{e}_\theta + u^2\boldsymbol{e}_\varphi$ on the sphere we have

$$\operatorname{div}\boldsymbol{u} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, u^1\right) + \frac{\partial u^2}{\partial\varphi}$$

(notice independence of R). Finally, for the Laplacian of a function on the sphere we have

$$\Delta f = \frac{1}{R^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\right).$$

From Example 6.25, comparing it with formula (36), we immediately see that on $\mathbb{R}^3$

$$\Delta = \Delta_R + \frac{1}{r^2}\,\Delta_A$$

where the first term

$$\Delta_R = \frac{1}{r^2}\frac{\partial}{\partial r}\,r^2\frac{\partial}{\partial r}$$

is called the *radial part* of the Laplace operator, and the second term is nothing but the Laplace operator on the sphere of radius r; the operator

$$\Delta_A = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\,\sin\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\,,$$

which is the Laplacian on the unit sphere, is known as the *angular part* of the Laplace operator on $\mathbb{R}^3$.

One can show that a similar decomposition holds for $\mathbb{R}^n$ for any n. We have

$$\Delta = \Delta_R + r^{-2}\Delta_A$$

where the 'radial part' is

$$\Delta_R f = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial f}{\partial r}\right) \quad \text{for } \mathbb{R}^n$$

and Δ_A is the Laplace operator for the unit sphere $S^{n-1} \subset \mathbb{R}^n$ with center at the origin in $\mathbb{R}^n$. (Partial differentiation in r is understood as the usual differentiation when the 'direction' $\boldsymbol{e} = r^{-1}\boldsymbol{r} \in S^{n-1}$ is fixed. Points of $\mathbb{R}^n$ can be treated as pairs $(r, \boldsymbol{e})$ where $\boldsymbol{r} = r\boldsymbol{e}$.)

Example 6.26. Let us find all functions annihilated by Δ (such functions are called *harmonic*) on $\mathbb{R}^n$ with the condition that they depend only on the radius r. For $f = f(r)$ we obtain $\Delta f = \Delta_R f$, and the condition $\Delta f = 0$ gives

$$\frac{d}{dr}\left(r^{n-1}\frac{df}{dr}\right) = 0$$

from where

$$r^{n-1}\frac{df}{dr} = a$$

for some constant a. Hence

$$f(r) = \int ar^{-n+1}dr = \begin{cases} a\,\frac{1}{-n+2}r^{-n+2} + b & \text{for } n \neq 2 \\ a\,\ln r + b & \text{for } n = 2\,, \end{cases}$$

which gives a two-dimensional space of solutions. Notice that besides trivial solutions (constants) we get an extra basis element, which is $f(r) = r$ on the line $\mathbb{R}$, and in higher dimensions is $f(r) = \ln r$ on $\mathbb{R}^2$ and $f(r) = 1/r^{n-2}$ on $\mathbb{R}^n$, $n > 2$. It has singularity at the origin (for $n > 1$).

It is not fitting here to go into the theory of harmonic functions in the general case when they can depend also on angular variables. (In particular, this involves the eigenvalue problem for the Laplace operator on the sphere.) This is done in textbooks on partial differential equations and equations of mathematical physics.

Now let us consider curl. Recall that divergence of a vector field $\boldsymbol{u}$ arises from considering the exterior differential of the flux form $\boldsymbol{u} \cdot d\boldsymbol{S}$. Similarly, curl is related with the circulation form. Consider the 1-form $\boldsymbol{u} \cdot d\boldsymbol{r}$. Its differential $d\,(\boldsymbol{u} \cdot d\boldsymbol{r})$ is a 2-form. In general, there is no relation with vector fields. However, if $2 = n - 1$, i.e., on a 3-space, we can view $d\,(\boldsymbol{u} \cdot d\boldsymbol{r})$ as the flux form of a new vector field. The corresponding linear map from vector fields to vector fields is precisely curl:

Definition 6.9. For vector fields on $\mathbb{R}^3$, the *curl* (or *rotor* or *rotation*) is the linear operator from vector fields to vector fields, notation: curl or rot, defined by the formula

$$d\,(\boldsymbol{u}\cdot d\boldsymbol{r}) = \operatorname{curl}\boldsymbol{u}\cdot d\boldsymbol{S}\,.$$

Using the formulas for circulation and flux forms it is not hard to obtain an explicit expression for curl. Denote by u_i (with lower indices) the components of the circulation 1-form of a vector field $\boldsymbol{u} = u^i\boldsymbol{e}_i$. We have $u_i = g_{ij}u^j$. Hence $d\,(\boldsymbol{u}\cdot d\boldsymbol{r}) = \sum_{i<j}(\partial_i u_j - \partial_j u_i)\,dx^i\wedge dx^j = (\partial_1 u_2 - \partial_2 u_1)\,dx^1\wedge dx^2 + (\partial_1 u_3 - \partial_3 u_1)\,dx^1\wedge dx^3 + (\partial_2 u_3 - \partial_3 u_2)\,dx^2\wedge dx^3$. Comparing it with the expression $\boldsymbol{w}\cdot d\boldsymbol{S} = \sqrt{g}(w^1dx^2\wedge dx^3 - w^2dx^1\wedge dx^3 + w^3dx^1\wedge dx^2)$ for the flux 2-form of a vector field $\boldsymbol{w}$, we arrive at the following statement.

Proposition 6.3. *In arbitrary coordinates on $\mathbb{R}^3$, the curl of a vector field $\boldsymbol{u}$ is given by the formula*

$$\begin{aligned}\operatorname{curl}\boldsymbol{u} =& \frac{1}{\sqrt{g}}\Big(\boldsymbol{e}_1(\partial_2 u_3 - \partial_3 u_2) - \boldsymbol{e}_2(\partial_1 u_3 - \partial_3 u_1) + \boldsymbol{e}_3(\partial_1 u_2 - \partial_2 u_1)\Big) = \\ & \frac{1}{\sqrt{g}}\begin{vmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ \partial_1 & \partial_2 & \partial_3\\ u_1 & u_2 & u_3\end{vmatrix}\,.\end{aligned}$$

Here $u_i = g_{ij}u^j$. (The symbolic determinant above is understood via the expansion in the first row, and is a mnemonic version of the preceding formula.)

Remark 6.8. The name "curl" is also used for $n > 3$, but then a "curl" of a vector field is just a 2-form $d\,(\boldsymbol{u}\cdot d\boldsymbol{r})$ with the coefficients $\partial_i u_j - \partial_j u_i$.

Example 6.27. In Cartesian coordinates on $\mathbb{R}^3$ we have

$$\operatorname{curl}\boldsymbol{u} = \begin{vmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ \partial_1 & \partial_2 & \partial_3\\ u_1 & u_2 & u_3\end{vmatrix}\,,$$

where $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3$.

Example 6.28. Consider on $\mathbb{R}^3$ the vector field $\boldsymbol{u} = \boldsymbol{\Omega}\times\boldsymbol{r}$ (the vector product), where $\boldsymbol{r}$ is the radius-vector and $\boldsymbol{\Omega}$ is a constant vector. It corresponds to a circular flow around the axis in the direction of $\boldsymbol{\Omega}$ through the origin. The vector $\boldsymbol{\Omega}$ has the meaning of the (constant) angular velocity. Let us find its curl. Working in Cartesian coordinates, we have $\boldsymbol{u} = \boldsymbol{e}_1(\Omega_2x_3 - \Omega_3x_2) - \boldsymbol{e}_2(\Omega_1x_3 - \Omega_3x_1) + \boldsymbol{e}_3(\Omega_1x_2 - \Omega_2x_1)$; hence for curl we obtain

$$\begin{aligned}\operatorname{curl}\boldsymbol{u} = &\begin{vmatrix}\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ \partial_1 & \partial_2 & \partial_3\\ \Omega_2x_3 - \Omega_3x_2 & -\Omega_1x_3 + \Omega_3x_1 & \Omega_1x_2 - \Omega_2x_1\end{vmatrix} = \\ & \boldsymbol{e}_1(\Omega_1 + \Omega_1) + \boldsymbol{e}_2(\Omega_2 + \Omega_2) + \boldsymbol{e}_3(\Omega_3 + \Omega_3) = 2\boldsymbol{\Omega}\,.\end{aligned}$$

Hence curl gives (double) the angular velocity of the flow. This answer explains in part the names "curl" and "rotation".

Example 6.29. Consider on $\mathbb{R}^3$ a 'central' field $\boldsymbol{u}$, i.e., one of the appearance $\boldsymbol{u} = f(r)\boldsymbol{r}$ where $\boldsymbol{r}$ is the radius-vector and $r = |\boldsymbol{r}|$. Let us find its curl. Apply spherical coordinates. We have $\boldsymbol{u} = f(r)r\boldsymbol{e}_r$ (since $\boldsymbol{e}_r = \boldsymbol{r}/r$). Recalling that in spherical coordinates $\sqrt{g} = r^2 \sin\theta$ and the matrix (g_{ij}) is diagonal with the entries $1, r^2, r^2 \sin^\theta$, we can write the formula for curl as

$$\operatorname{curl} \boldsymbol{u} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_\varphi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ u^1 & r^2\, u^2 & r^2 \sin^2\theta\, u^3 \end{vmatrix}$$

if $\boldsymbol{u} = u^1\boldsymbol{e}_r + u^2\boldsymbol{e}_\theta + u^3\boldsymbol{e}_\varphi$. Hence, in our particular case,

$$\operatorname{curl}\left(f(r)\boldsymbol{r}\right) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & \boldsymbol{e}_\theta & \boldsymbol{e}_\varphi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\varphi \\ f(r)r & 0 & 0 \end{vmatrix} = 0\,.$$

Remark 6.9. Developing Example 6.28, one can show, by considering the Taylor expansion of a vector field $\boldsymbol{u}$ on $\mathbb{R}^n$ (viewed as a map $\boldsymbol{u}\colon \mathbb{R}^n \to \mathbb{R}^n$) near an arbitrary point and retaining only the constant and the first-order terms, that near any point the flow corresponding to the field is, approximately, the composition of a constant shift (which corresponds to the zero-order term), a stretching/compressing in three orthogonal directions, and a rotation. The latter two transformations arise from the symmetric and antisymmetric parts of the linear term (i.e., the differential of the map at the given point), respectively. The divergence is the trace of the symmetric part and it is responsible for the change of volume as the result of the stretching/compression. The "curl" (which is in general just a 2-form and specified by an antisymmetric matrix) is the antisymmetric part and it corresponds to the rotation.

There is a convenient description of the divergence, Laplacian and curl using the *nabla* notation introduced above. Notice that *nabla* has a twofold nature: it is a differential operator and a vector. One can try to use the 'vector nature' of $\boldsymbol{\nabla}$ and apply to it vector operations such as the scalar product (in $\mathbb{R}^n$, for any n) and the vector product (in $\mathbb{R}^3$). Doing so, one, however, should be careful and not forget that the 'components' of the vector $\boldsymbol{\nabla}$ are operators of differentiation, hence it should be clarified to which object they will be applied. It follows that familiar features such as commutativity could be lost: consider, for example the scalar products $\boldsymbol{\nabla} \cdot \boldsymbol{u}$ and $\boldsymbol{u} \cdot \boldsymbol{\nabla}$ where $\boldsymbol{u}$ is an arbitrary vector field on $\mathbb{R}^n$. (The 'dot' notation for the scalar product is convenient here.) Are they the same?

The rule is as follows: the operation of differentiation is applied to all what is to the right of it (but not, to the left), unless otherwise prescribed explicitly. With this understanding, the following statement holds.

Proposition 6.4. *For any vector field $\boldsymbol{u}$ on $\mathbb{R}^n$, the scalar product $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ is nothing but the divergence of $\boldsymbol{u}$:*

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \operatorname{div}\boldsymbol{u}\,. \tag{37}$$

The scalar product $\boldsymbol{u}\cdot\boldsymbol{\nabla}$ is the operator of differentiation along $\boldsymbol{u}$:

$$\boldsymbol{u}\cdot\boldsymbol{\nabla} = \partial_{\boldsymbol{u}} \tag{38}$$

(which by definition acts on any function as $\langle df, \boldsymbol{u}\rangle$).

Proof. To prove (37), use Cartesian coordinates. We have $\boldsymbol{\nabla} = \sum \boldsymbol{e}_i\partial_i$; if $\boldsymbol{u} = u^i\boldsymbol{e}_i$, then $\boldsymbol{\nabla}\cdot\boldsymbol{u} = \partial_i u^i = \operatorname{div}\boldsymbol{u}$ (we took the sum of 'products' of components and used the rule that differentiation applies to what is to the right of it). A calculation in any other coordinates, non-Cartesian, will of course give the same result but take more time. To prove (38), we can work without coordinates. Since $\boldsymbol{u}\cdot\boldsymbol{\nabla}$ is going to be a differential operator ($\boldsymbol{\nabla}$ applies to what will appear at the right of it and *not* applies to $\boldsymbol{u}$), we have to check its value on an arbitrary 'test' function f. We have $(\boldsymbol{u}\cdot\boldsymbol{\nabla})(f) = \boldsymbol{u}\cdot\boldsymbol{\nabla}f = \boldsymbol{u}\cdot\operatorname{grad}f = \operatorname{grad}f\cdot\boldsymbol{u} = \langle df, \boldsymbol{u}\rangle = \partial_{\boldsymbol{u}}f$. We used the definition of $\operatorname{grad}f$ as the vector field corresponding to the 1-form df. □

(We see that $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ and $\boldsymbol{u}\cdot\boldsymbol{\nabla}$ are not only the same, but objects of different nature: a function and a differential operator.)

In $\mathbb{R}^3$ one can consider the vector ("cross") product.

Proposition 6.5. *For an arbitrary vector field $\boldsymbol{u}$ on $\mathbb{R}^3$ we have*

$$\boldsymbol{\nabla}\times\boldsymbol{u} = \operatorname{curl}\boldsymbol{u}\,. \tag{39}$$

Proof. In Cartesian coordinates the cross product $\boldsymbol{\nabla}\times\boldsymbol{u}$ is given by the symbolic determinant

$$\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

(where it is understood that partial derivatives act on the components of $\boldsymbol{u}$; here $u_i = u^i$). It is precisely the same determinant that expresses $\operatorname{curl}\boldsymbol{u}$. □

Consider, finally, the scalar product of $\boldsymbol{\nabla}$ with itself.

Proposition 6.6. *The scalar square of $\boldsymbol{\nabla}$ is the Laplace operator:*

$$\boldsymbol{\nabla}^2 = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} = \Delta\,.$$

Proof. In Cartesian coordinates $\boldsymbol{\nabla} = \sum \boldsymbol{e}_i\partial_i$ and $\boldsymbol{\nabla}^2 = \sum(\partial_i)^2 = \Delta$. □

Notation $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ and $\boldsymbol{\nabla}\times\boldsymbol{u}$ for the divergence and curl of $\boldsymbol{u}$, and $\boldsymbol{\nabla}^2$ for the Laplace operator, is especially popular in physical and engineering literature. Historically, it was the first condensed language for vector calculus, before the introduction of theory of differential forms. Working with $\boldsymbol{\nabla}$ and using its 'vector properties', with the appropriate care, can be very handy.

Example 6.30. Deduce the identity:

$$\operatorname{div}(f\boldsymbol{u}) = \operatorname{grad} f\cdot\boldsymbol{u} + f\operatorname{div}\boldsymbol{u}\,.$$

Solution: $\operatorname{div}(f\boldsymbol{u}) = \boldsymbol{\nabla}\cdot(f\boldsymbol{u})$; it is not possible to simply take f out, as $\boldsymbol{\nabla}$ has to act on it. Partial derivatives that are 'inside' of $\boldsymbol{\nabla}$ obey the product rule. A convenient way of taking it into account is to write $\boldsymbol{\nabla} = \boldsymbol{\nabla}^{(f)} + \boldsymbol{\nabla}^{(\boldsymbol{u})}$, where in this provisional notation a label such as f or $\boldsymbol{u}$ attached to $\boldsymbol{\nabla}$ means that differentiation will be applied the the specified object and only to it. After attaching these labels we can deal with *nabla* as with a usual vector, moving it around as we please. Hence we have $\operatorname{div}(f\boldsymbol{u}) = (\boldsymbol{\nabla}^{(f)} + \boldsymbol{\nabla}^{(\boldsymbol{u})})\cdot(f\boldsymbol{u}) = \boldsymbol{\nabla}^{(f)}\cdot(f\boldsymbol{u}) + \boldsymbol{\nabla}^{(\boldsymbol{u})}\cdot(f\boldsymbol{u}) = \boldsymbol{\nabla}^{(f)}f\cdot\boldsymbol{u} + f\boldsymbol{\nabla}^{(\boldsymbol{u})}\cdot\boldsymbol{u} = \boldsymbol{\nabla}f\cdot\boldsymbol{u} + f\boldsymbol{\nabla}\cdot\boldsymbol{u} = \operatorname{grad} f\cdot\boldsymbol{u} + f\operatorname{div}\boldsymbol{u}$.

To summarize our discussion of divergence and curl, let us use diagrams to represent various vector spaces and linear transformations between them. Let $\Omega^k(\mathbb{R}^n)$ be the space of k-forms on $\mathbb{R}^n$, as usual, and denote the spaces of (infinitely differentiable) functions and vector fields on $\mathbb{R}^n$ by $\operatorname{Fun}(\mathbb{R}^n)$ and $\operatorname{Vect}(\mathbb{R}^n)$, respectively. Then we have two commutative diagrams:

$$\begin{tikzcd}
\Omega^0(\mathbb{R}^n) \arrow[r, "d"] & \Omega^1(\mathbb{R}^n) \arrow[r, "d"] & \Omega^2(\mathbb{R}^n) \arrow[r, "d"] & \dots \arrow[r, "d"] & \Omega^{n-1}(\mathbb{R}^n) \arrow[r, "d"] & \Omega^n(\mathbb{R}^n) \\
\operatorname{Fun}(\mathbb{R}^n) \arrow[u, "="] \arrow[r, "\operatorname{grad}"] & \operatorname{Vect}(\mathbb{R}^n) \arrow[u, "\cdot d\boldsymbol{r}"'] & & & \operatorname{Vect}(\mathbb{R}^n) \arrow[u, "\cdot d\boldsymbol{S}"] \arrow[r, "\operatorname{div}"] & \operatorname{Fun}(\mathbb{R}^n) \arrow[u, "\cdot dV"']
\end{tikzcd}$$

for $\mathbb{R}^n$, if $n \neq 3$, and

$$\begin{tikzcd}
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] & \Omega^3(\mathbb{R}^3) \\
\operatorname{Fun}(\mathbb{R}^3) \arrow[u, "="] \arrow[r, "\operatorname{grad}"] & \operatorname{Vect}(\mathbb{R}^3) \arrow[u, "\cdot d\boldsymbol{r}"'] \arrow[r, "\operatorname{curl}"] & \operatorname{Vect}(\mathbb{R}^3) \arrow[u, "\cdot d\boldsymbol{S}"] \arrow[r, "\operatorname{div}"] & \operatorname{Fun}(\mathbb{R}^3) \arrow[u, "\cdot dV"']
\end{tikzcd}$$

for $\mathbb{R}^3$.

("Commutative" means that if it is possible to get from one point at the diagram to another point travelling along the arrows by two different ways, then the compositions of maps along these ways will be the same. For example, above one can start from $\boldsymbol{u} \in \operatorname{Vect}(\mathbb{R}^n)$, take the dot product with $d\boldsymbol{S}$ getting the $(n-1)$-form $\boldsymbol{u}\cdot d\boldsymbol{S} \in \Omega^{n-1}(\mathbb{R}^n)$ and apply d, arriving at $d(\boldsymbol{u}\cdot d\boldsymbol{S}) \in \Omega^n(\mathbb{R}^n)$. Or one take the divergence, obtaining $\operatorname{div}\boldsymbol{u} \in$

$\mathrm{Fun}(\mathbb{R}^n)$, and multiply by the volume form, obtaining $(\operatorname{div}\boldsymbol{u})\,dV \in \Omega^n(\mathbb{R}^n)$. By definition of divergence, the results coincide.)

For $\mathbb{R}^n$, $n \neq 3$, instead of curl taking values in vector fields, we can consider a map $\mathrm{Vect}(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n)$, the composition of the map $\boldsymbol{u} \mapsto \boldsymbol{u}\cdot d\boldsymbol{r}$ with d.

The identity $d^2 = 0$ is equivalent in $\mathbb{R}^3$ to

$$\operatorname{curl}\operatorname{grad} f = 0, \quad \operatorname{div}\operatorname{curl}\boldsymbol{u} = 0$$

for all functions f and vector fields $\boldsymbol{u}$. Vector fields of the appearance $\boldsymbol{E} = \operatorname{grad} f$ and $\boldsymbol{H} = \operatorname{curl}\boldsymbol{A}$ are called *potential* and *solenoidal*, respectively. (Then the function f is called a 'scalar potential' for $\boldsymbol{E}$, and the vector field $\boldsymbol{A}$, a 'vector potential' for $\boldsymbol{H}$.) They correspond to exact 1- and 2-forms on $\mathbb{R}^3$. Vector fields satisfying $\operatorname{curl}\boldsymbol{E} = 0$ and $\operatorname{div}\boldsymbol{H} = 0$ are called *irrotational* and *divergence-free*, respectively. They correspond to closed 1- and 2-forms. Each potential field is irrotational, and each solenoidal field, divergence-free. The converse is true in simple domains, such as the whole space $\mathbb{R}^3$, but may be wrong in the domains "with holes". (See examples of closed forms that are not exact.)

What about other possible compositions of grad, div and curl? We know that $\operatorname{div}\operatorname{grad} = \boldsymbol{\nabla}^2$ gives the Laplace operator Δ on functions. Consider grad div and curl curl. Both are operators on vector fields. It turns out that together they combine into an analog of the Laplacian, this time for vector fields on $\mathbb{R}^3$. Recall the vector identity $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$. Let us make a formal calculation with the operator *nabla*:

$$\operatorname{curl}\operatorname{curl}\boldsymbol{u} = \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = (\boldsymbol{\nabla}\cdot\boldsymbol{u})\boldsymbol{\nabla}^{(\boldsymbol{u})} - (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{u} =$$
$$\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}^2\boldsymbol{u} = \operatorname{grad}\operatorname{div}\boldsymbol{u} - \boldsymbol{\nabla}^2\boldsymbol{u}$$

(we put a label on $\boldsymbol{\nabla}$ appearing to the right of $\boldsymbol{u}$ to show that it should act on $\boldsymbol{u}$). Hence $\Delta = \boldsymbol{\nabla}^2$ acting on vector fields can be *defined* using the above formal equality.

Definition 6.10. The *Laplace operator* acting on vector fields on $\mathbb{R}^3$, notation: Δ or $\boldsymbol{\nabla}^2$ (the same as for functions), is defined by

$$\Delta\boldsymbol{u} = \operatorname{grad}\operatorname{div}\boldsymbol{u} - \operatorname{curl}\operatorname{curl}\boldsymbol{u}\,. \tag{40}$$

Example 6.31. Consider Cartesian coordinates. We have $\boldsymbol{\nabla} = \boldsymbol{e}_1\partial_1 + \boldsymbol{e}_2\partial_2 + \boldsymbol{e}_3\partial_3$; if applied to a vector field, a partial derivative acts only on the components (since the basis vectors $\boldsymbol{e}_i$ are constant). Therefore the formal calculation above can be fully justified with the understanding that partial derivatives are applied to components. As the result we have

$$\Delta\boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) - \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = (\Delta u_1)\boldsymbol{e}_1 + (\Delta u_2)\boldsymbol{e}_2 + (\Delta u_3)\boldsymbol{e}_3$$

if $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3$. In other words, <u>in Cartesian coordinates</u> Δ on vector fields simply acts as the usual Laplacian on functions applied to each of the components.

In non-Cartesian coordinates the explicit expression for Δ on vector fields will be more complicated, and it will not reduce to the Laplacian for functions acting componentwise. One should use equation (40). As the reader may suspect, behind all this is a more general theory of Laplace operators acting on differential k-forms, for each k, and valid for any dimension n. (In $\mathbb{R}^3$ there are just two basically different cases: $k=0$, i.e., functions; and $k=1$ and $k=2=3-1$, which are essentially the same, both corresponding to vector fields.)

6.4 Classical integral theorems

6.4.1 Ostrogradski–Gauß theorem

The specialization of the general Stokes theorem for the flux form $\boldsymbol{F}\cdot d\boldsymbol{S}$ and the circulation form $\boldsymbol{X}\cdot d\boldsymbol{r}$ gives two classical integral theorems traditionally associated with the names of Ostrogradski, Gauß, and Stokes.

Recall that for any oriented surface S of dimension $n-1$ in $\mathbb{R}^n$ or an $(n-1)$-chain the *flux* of a vector field $\boldsymbol{F}$ through S is defined as the integral of the flux form $\boldsymbol{F}\cdot d\boldsymbol{S}$ over S:

$$\int_S \boldsymbol{F}\cdot d\boldsymbol{S}.$$

The general Stokes theorem and the definition of $\operatorname{div}\boldsymbol{F}$ immediately imply

Theorem 6.2 (Ostrogradski–Gauß theorem). *The flux of a vector field $\boldsymbol{F}$ defined on a bounded domain $D\subset\mathbb{R}^n$ through the boundary of D, equals the volume integral of the divergence of $\boldsymbol{F}$:*

$$\oint_{\partial D} \boldsymbol{F}\cdot d\boldsymbol{S} = \int_D \operatorname{div}\boldsymbol{F}\,dV. \tag{41}$$

or

$$\oint_{\partial D} (\boldsymbol{F}\cdot\boldsymbol{n})dS = \int_D \operatorname{div}\boldsymbol{F}\,dV \tag{42}$$

where $\boldsymbol{n}$ is the unit normal to ∂D.

Passing from (41) to (42) takes into account Proposition 6.2.

Example 6.32. Consider the flux of the vector field

$$\boldsymbol{E} = -\frac{\boldsymbol{r}}{r^3} \tag{43}$$

on $\mathbb{R}^3$ (the "Coulomb force"), through the sphere of radius R oriented by the outward normal. The Ostrogradski–Gauß theorem is not applicable to $\boldsymbol{E}$ because $\boldsymbol{E}$ is not defined at the origin O. Let us evaluate the flux directly. Indeed, $\boldsymbol{r}$ points in the direction of the outward normal and we have $(\boldsymbol{E}\cdot\boldsymbol{n})\,dS = -RR^{-3}\,dS = -R^{-2}\,dS$ (as $r=R$ on the sphere). Hence

$$\oint_{S_R} \boldsymbol{E}\cdot d\boldsymbol{S} = \oint_{S_R} (\boldsymbol{E}\cdot\boldsymbol{n})\,dS = -R^{-2}\oint_{S_R} dS = -R^{-2}\operatorname{area} S_R = -4\pi. \tag{44}$$

We see that remarkably the flux does not depend on radius. The explanation is that the form $-r^{-3}\boldsymbol{r}\cdot d\boldsymbol{S}$ is closed, or, equivalently, that $\operatorname{div}(-r^{-3}\boldsymbol{r}) = 0$, for $\boldsymbol{r} \neq 0$, in $\mathbb{R}^3$. Therefore for two concentric spheres of radii R and R' the difference of the fluxes of $\boldsymbol{E}$, which is the flux through the boundary of the region between the two spheres, can be now found using the Ostrogradski–Gauß theorem, and it vanishes because $\operatorname{div}\boldsymbol{E}$ vanishes.

Example 6.33. The Ostrogradski–Gauß theorem can be applied to calculating the flux in the previous Example if we use a trick. Notice that on the surface of the sphere $r = R$, hence $\boldsymbol{E} = -\frac{\boldsymbol{r}}{r^3}$ can be replaced by the field $\boldsymbol{E}' = -\frac{\boldsymbol{r}}{R^3}$, which well-defined at the origin. We have $\operatorname{div}(-\frac{\boldsymbol{r}}{R^3}) = -R^{-3}\operatorname{div}\boldsymbol{r} = -3R^{-3}$, as $\operatorname{div}\boldsymbol{r}$ in $\mathbb{R}^n$ is n. Hence, by the Ostrogradski–Gauß theorem,

$$\oint_{S_R} \boldsymbol{E}\cdot d\boldsymbol{S} = \oint_{S_R} \boldsymbol{E}'\cdot d\boldsymbol{S} = \int_{B_R} \operatorname{div}\boldsymbol{E}' = \int_{B_R} (-3R^{-3}) = -3R^{-3}\operatorname{vol} B_R = -3R^{-3}\frac{4}{3}\pi R^3 = -4\pi\,,$$

agreeing, of course, with the result of the direct calculation. Here B_R stands for the ball of radius R, so $S_R = \partial B_R$.

From the Ostrogradski–Gauß theorem follows an "integral definition" of the divergence: at any point $\boldsymbol{x}_0 \in \mathbb{R}^n$

$$\operatorname{div}\boldsymbol{F}(\boldsymbol{x}_0) = \lim_{D\to\boldsymbol{x}_0} \frac{\oint_{\partial D}\boldsymbol{F}\cdot d\boldsymbol{S}}{\operatorname{vol} D}. \tag{45}$$

Here $\boldsymbol{x}_0 \in D$ and $D \to \boldsymbol{x}_0$ means that the domain D "shrinks" to a point $\boldsymbol{x}_0$. Thus the divergence at $\boldsymbol{x}_0$ measures the intensity of a "source" of the flow at the point $\boldsymbol{x}_0$. If it is negative, the "source" is actually a "sink". All these concepts come from the hydrodynamical interpretation.

6.4.2 The classical Stokes theorem

Another statement following from the general Stokes theorem and which gave to it the name, is the "classical Stokes theorem". Unlike the Ostrogradski–Gauß theorem, it is stated only for $\mathbb{R}^3$.

Theorem 6.3 (Classical Stokes theorem). *The circulation of a vector field over the boundary of any oriented surface or* 2*-chain S in $\mathbb{R}^3$ equals the flux of the curl of $\boldsymbol{X}$ through S:*

$$\oint_{\partial S} \boldsymbol{X}\cdot d\boldsymbol{r} = \int_S \operatorname{curl}\boldsymbol{X}\cdot d\boldsymbol{S}.$$

It immediately follows from the definition of curl: $\operatorname{curl}\boldsymbol{X}\cdot d\boldsymbol{S} = d(\boldsymbol{X}\cdot d\boldsymbol{r})$.

Like in the Ostrogradski–Gauß theorem, the flux in the classical Stokes theorem can be re-written as an integral of the first kind, and we have

$$\oint_{\partial S} \boldsymbol{X}\cdot d\boldsymbol{r} = \int_S (\operatorname{curl}\boldsymbol{X}\cdot\boldsymbol{n})\, dS\,,$$

where $\boldsymbol{n}$ is the unit normal for S.

Theorem 6.3 gives rise to an "integral definition" of curl similar to that of divergence above: at any point $\boldsymbol{x}_0 \in \mathbb{R}^3$ and any unit vector $\boldsymbol{n}$

$$\operatorname{curl}\boldsymbol{X}(\boldsymbol{x}_0)\cdot\boldsymbol{n} = \lim_{S\to\boldsymbol{x}_0} \frac{\oint_{\partial S}\boldsymbol{X}\cdot d\boldsymbol{r}}{\operatorname{area} S}. \tag{46}$$

Here S is a bounded surface such that $\boldsymbol{x}_0 \in S$ and $\boldsymbol{n}$ is the unit normal to S at $\boldsymbol{x}_0$. Here the limit $S \to \boldsymbol{x}_0$ means that the surface S "shrinks" to the point $\boldsymbol{x}_0$. Therefore the projection of $\operatorname{curl}\boldsymbol{X}$ onto a given direction $\boldsymbol{n}$ measures the circulation of the flow of $\boldsymbol{X}$ around $\boldsymbol{x}_0$ in the surface normal to $\boldsymbol{n}$, relative to the area of the surface.