VECTOR CALCULUS 214

Fall 2005. COURSEWORK

Deadline: Friday afternoon, 11 November 2005

Answer ALL questions. Each question is worth 4 marks

WRITE SOLUTIONS IN THE PROVIDED SPACES.
IF NECESSARY, USE THE OTHER SIDE OF THE PAGE

STUDENT'S NAME:

Problem 1. Consider the form

$$\omega = \frac{dx \wedge dy}{(1 + x^2 + y^2)^2}\,.$$

Find its pullback $F^*\omega$ w.r.t. the map $F\colon\ (u, v) \mapsto (x, y)$ where

$$x = \frac{u}{u^2 + v^2}\,, \quad y = -\frac{v}{u^2 + v^2}\,.$$

Problem 2. Calculate the exterior differentials of the following forms:

(a) $\omega = r^3(\cos 4\theta\, dr - r \sin 4\theta\, d\theta)$

(b) $A = e^{i(\boldsymbol{k},\boldsymbol{x})}\,(A_1\, dx + A_2\, dy + A_3\, dz)$

(c) $B = e^{i(\boldsymbol{k},\boldsymbol{x})}\,(B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dx \wedge dy)$

Here (A_1, A_2, A_3), (B_1, B_2, B_3), $\boldsymbol{k} = (k_1, k_2, k_3)$ are arbitrary constant vectors in $\mathbb{R}^3$ and $(\boldsymbol{k}, \boldsymbol{x}) = k_1 x + k_2 y + k_3 z$ is the Euclidean scalar product of $\boldsymbol{k}$ and the radius-vector $\boldsymbol{x} = (x, y, z)$.

Problem 3. Consider the following form on $\mathbb{R}^n \setminus \{0\}$:

$$\begin{aligned}\sigma =& r^\alpha \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \ldots \wedge dx^{i-1} dx^{i+1} \wedge \ldots \wedge dx^n = \\ & r^\alpha \Big(x^1 dx^2 \wedge \ldots \wedge dx^n - x^2 dx^1 \wedge dx^3 \wedge \ldots \wedge dx^n + \ldots + (-1)^{n-1} x^n dx^1 \wedge \ldots \wedge dx^{n-1} \Big) .\end{aligned}$$

Here $\alpha \in \mathbb{R}$ is a parameter, and $r = \sqrt{(x^1)^2 + \ldots + (x^n)^2}$. Find $d\sigma$ and determine for which values of α it vanishes.

Problem 4. Given a map $F\colon U \to \mathbb{R}^2$ where $U = \{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}$:

$$F\colon \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{2}\ln\left(x^2 + y^2\right) \\ \arctan \dfrac{y}{x} \end{pmatrix}$$

Find the matrix of dF w.r.t. the standard basis $\boldsymbol{e}_1 = (1, 0), \boldsymbol{e}_2 = (0, 1)$.

Problem 5. In this problem you will have to recall properties of the determinant and trace of square matrices.

(a) Check the following identity (known as the *Liouville formula*):

$$\det e^{A} = e^{\operatorname{tr} A}$$

for all $n \times n$ matrices. Here the exponential of matrices is defined as the sum of the power series $E + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$. (You may assume that the matrix is diagonalizable.)

(b) Let $A = A(t)$ be an arbitrary parametrized curve in the space of $n \times n$ matrices such that $A(0) = E$ (the identity matrix). Let $\dot{A} = \dot{A}(t)$ denote its velocity. Show, by using the standard expansion of the determinant, $\det A = \sum_{\sigma} \operatorname{sgn} \sigma \cdot A_{1\sigma(1)} \dots A_{n\sigma(n)}$ (sum over all permutations), that

$$\left.\frac{d}{dt}\right|_{t=0} \det A(t) = \operatorname{tr} \dot{A}(0)\,.$$

Hint: differentiate using the product rule and recall the value of $A(t)$ at $t = 0$.

(c) Apply the results of parts **(a)** and **(b)** to find the vector space $T_E SL(n)$, the tangent space at E for the *special linear group* $SL(n)$. Recall that $SL(n)$ is specified by the equation $\det A = 1$ in the space of all $n \times n$ matrices. *Hint:* to know what to look for, you may first consider matrices 2×2.