Introduction to Topology (Maths 353)

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A. List of statements

X, Y, ... denote topological spaces (unless stated otherwise), and $\mathcal{O}_X$, $\mathcal{O}_Y$, ..., their respective topologies.

1 Topological spaces and continuous maps

Theorem. *For a map of metric spaces $f\colon X \to Y$, the continuity of f at a point $x_0 \in X$ is equivalent to the following condition: for every neighborhood V of $y_0 = f(x_0)$ there is a neighborhood U of x_0 such that $f(U) \subset V$.*

Theorem. *A map of topological spaces $f\colon X \to Y$ is continuous at all points $x \in X$ if and only if for every open set $V \subset Y$ its preimage $f^{-1}(V)$ is open in X.*

Theorem. *The composition of continuous maps is continuous. For each topological space the identity map is continuous.*

Theorem. *A map between two topological spaces is continuous if and only if it is continuous at every point.*

Theorem. *Homeomorphism is an equivalence relation for topological spaces.*

2 Topological constructions

2.1 Induced topology and subspaces

Theorem. *Let $f\colon X \to Y$ be a map where X is a set, Y is a topological space. Denote $f^*\mathcal{O}_Y := \{U \subset X \,|\, U = f^{-1}(V) \text{ where } V \in \mathcal{O}_Y\}$. Then: (1) $f^*\mathcal{O}_Y$ is a topology on X, called the* induced topology *(more precisely, the* topology induced by f*); (2) $f\colon (X, f^*\mathcal{O}_Y) \to (Y, \mathcal{O}_Y)$ is continuous; (3) $f^*\mathcal{O}_Y$ is the smallest topology on X with this property (i.e., every topology w.r.t. which the map f is continuous contains $f^*\mathcal{O}_Y$).*

Theorem. *Let $A \subset X$ be a subspace in X. A map $f\colon Z \to A$ is continuous if and only if the map $i \circ f\colon Z \to X$ is continuous. (Here $i\colon A \to X$ stands for the inclusion).*

2.2 Coinduced topology and identification spaces

Theorem. *Let* $f: X \to Y$ *be a map where* X *is a topological space,* Y *is a set. Denote* $f_*\mathcal{O}_X := \{V \subset Y \mid f^{-1}(V) \in \mathcal{O}_X\}$*. Then: (1)* $f_*\mathcal{O}_X$ *is a topology on* Y*, called the* coinduced topology *(more precisely, the* topology coinduced by f*); (2)* $f: (X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ *is continuous; (3)* $f_*\mathcal{O}_X$ *is the largest topology on* Y *with this property (i.e., every topology w.r.t. which the map* f *is continuous is contained in* $f_*\mathcal{O}_X$*).*

Theorem. *For an identification space* X/R*, a map* $f: X/R \to Y$ *is continuous if and only if* $f \circ p: X \to Y$ *is continuous. (Here* $p: X \to X/R$ *stands for the projection)*

2.3 Product topology

Theorem. *(1)* $\mathcal{B}_{X\times Y} := \{U \times V \mid U \in \mathcal{O}_X, V \in \mathcal{O}_Y\}$ *is a base, the topology generated by this base is called the* product topology*, notation:* $\mathcal{O}_{X\times Y}$*; (2) the maps* $p_1: X \times Y \to X$ *and* $p_2: X \times Y \to Y$ *are continuous w.r.t. the product topology on* $X \times Y$*; (3) the product topology is the smallest topology on* $X \times Y$ *with such property.*

Theorem. *A map* $f: Z \to X \times Y$ *is continuous* (w.r.t. the product topology) *if and only if the maps* $f_1 = p_1 \circ f: Z \to X$ *and* $f_2 = p_2 \circ f: Z \to Y$ *are continuous.*

Theorem. *Given an arbitrary topological space* Z *and continuous maps* $f_1: Z \to X$ *and* $f_2: Z \to Y$*, there exists a unique continuous map* $f: Z \to X \times Y$ *such that* $f_1 = p_1 \circ f$*,* $f_2 = p_2 \circ f$*.*

3 Fundamental topological properties

3.1 Closed sets

Theorem. *A map* $f: X \to Y$ *is continuous if and only if for every closed set* $C \subset Y$ *its preimage* $f^{-1}(C)$ *is closed in* X*.*

3.2 Hausdorff property

Theorem. *Every metric space is Hausdorff.* (In particular, $\mathbb{R}^n$ is Hausdorff.)

Proposition. *Every point in a Hausdorff space is closed.*

Theorem. *Every subspace of a Hausdorff topological space is Hausdorff.*

Corollary. *Every subspace of* $\mathbb{R}^N$ *is Hausdorff.*

Theorem. *If* X*,* Y *are Hausdorff, then* $X \times Y$ *is Hausdorff.*

3.3 Compactness

Theorem. *If X is compact, $f\colon X \to Y$ is continuous, then $f(X)$ is compact.*

Theorem. *If X, Y are compact, then $X \times Y$ is compact.* (Proof required for MSc only)

Theorem. *A closed subspace of a compact space is compact.*

Lemma. *Let X be Hausdorff, $K \subset X$ be a compact subspace, and $a \notin K$. Then there are open sets U and V such that $a \in U$, $K \subset V$, and $U \cap V = \varnothing$.*

Theorem. *Let X be Hausdorff and $K \subset X$ be a compact subspace. Then K is a closed subset.*

Theorem (Homeomorphism Theorem). *If X is compact, Y is Hausdorff, $f\colon X \to Y$ is continuous and invertible, then f is a homeomorphism.* (In other words, the inverse map will be automatically continuous.)

Corollary. *If X is compact, Y is Hausdorff, $f\colon X \to Y$ is continuous and invjective (one-to-one), then f is a homeomorphism of X onto the subspace $f(X) \subset Y$.*

Theorem. *If a metric space is compact, then it is bounded.*

Lemma (Heine–Borel Lemma). *A cube in $\mathbb{R}^n$ is compact.* (Proof required for MSc only)

Theorem (Heine–Borel Theorem). *A subspace in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

3.4 Connectedness and path-connectedness

Theorem. *If X is connected, $f\colon X \to Y$ is continuous, then $f(X)$ is connected.*

Theorem. *If X, Y are connected, then $X \times Y$ is connected.*
(Proof required for MSc only)

Lemma. *Every segment $[a, b]$ is connected.* (Proof required for MSc only)

Theorem. *If X is path-connected, then X is connected.*

Theorem. *If X, Y are path-connected, then $X \times Y$ is path-connected.*

Theorem. *If X is path-connected, $f\colon X \to Y$ is continuous, then $f(X)$ is path-connected.*

Theorem. *Every connected open subspace of a Euclidean space $U \subset \mathbb{R}^n$ is path-connected.*

Theorem. *If every point of X has a path-connected open neighborhood and X is connected, then X is path-connected.*

4 Manifolds and surfaces

Theorem (Classification Theorem for Closed Surfaces). *Every closed surface is homeomorphic to one of the standard surfaces: the sphere S^2, the sphere with g handles H^2_g, or the sphere with μ Möbius strips M^2_μ.*

5 Simplicial complexes and Euler characteristic

Theorem (Topological Invariance of Euler Characteristic). *For simplicial complexes K and L, if $|K| \cong |L|$, then $\chi(K) = \chi(L)$.* **[No proof]**

Theorem (Excision Formula). *Suppose K, L are subcomplexes of a simplicial complex N so that $N = K \cup L$. Then*

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L).$$

B. Extra material for MSc students

1. Compactness:

Theorem 1. If X, Y are compact, then $X \times Y$ is compact.

Proof. Suppose X, Y are compact. To prove that the product space $X \times Y$ is compact, consider an arbitrary cover of $X \times Y$ by open sets of the form $U_\alpha \times V_\alpha$, where $U_\alpha \in \mathcal{O}_X$, $V_\alpha \in \mathcal{O}_Y$. Let $x \in X$ be an arbitrary point of X. Consider the subspace $\{x\} \times Y \subset X \times Y$. It is compact (as homeomorphic to Y), hence

$$\{x\} \times Y \subset U_{\alpha_1(x)} \times V_{\alpha_1(x)} \cup \ldots \cup U_{\alpha_N(x)} \times V_{\alpha_N(x)}.$$

Consider the intersection $U^x := U_{\alpha_1(x)} \cap \ldots \cap U_{\alpha_N(x)}$. Notice that it is an open subset of X. We have, clearly,

$$U^x \times Y \subset U_{\alpha_1(x)} \times V_{\alpha_1(x)} \cup \ldots \cup U_{\alpha_N(x)} \times V_{\alpha_N(x)}.$$

On the other hand, the collection of all U^x (for all $x \in X$) is an open cover of X. Since X is compact, there is a finite subcover: $X = U^{x_1} \cup \ldots \cup U^{x_{N'}}$. It follows that

$$\begin{aligned} X \times Y = U^{x_1} \times Y \cup \ldots \cup U^{x_{N'}} \times Y \subset \\ \Big(U_{\alpha_1(x_1)} \times V_{\alpha_1(x_1)} \cup \ldots \cup U_{\alpha_N(x_1)} \times V_{\alpha_N(x_1)}\Big) \cup \ldots \cup \\ \Big(U_{\alpha_1(x_{N'})} \times V_{\alpha_1(x_{N'})} \cup \ldots \cup U_{\alpha_N(x_{N'})} \times V_{\alpha_N(x_{N'})}\Big). \end{aligned}$$

□

2. Connectedness and path-connectedness:

You should use the book by Armstrong, *Basis Topology*, Section 3.5. There you can find the proofs of the following statements:

Theorem. *If X, Y are connected, then $X \times Y$ is connected.*

Theorem. *Real line is connected. Every segment $[a, b]$ is connected.*