§1 Topological spaces and continuous maps

Problem 1. The list of all possible topologies is as follows. The smallest topology: $\{\varnothing, X\}$ (the indiscrete topology). Then $\{\varnothing, X, \{a\}\}$ and $\{\varnothing, X, \{b\}\}$. Then $\{\varnothing, X, \{a\}, \{b\}\}$ (all subsets, the discrete topology). It is immediately checked that in each of these cases the axioms T1, T2, T3 are satisfied.

Problem 2. The set $U \subset \mathbb{R}^n$ is *open* if for every point $x \in U$ there is an ε-ball $B_\varepsilon(x)$ with center at x such that $B_\varepsilon(x) \subset U$. Check T1, T2, T3. T1 is trivial: for $\varnothing$ there are no points, hence no condition; for X, take, e.g., $B_1(x)$ for each point. Now T2. Suppose we have two open sets U, V. Let $x \in U \cap V$; we have $B_\varepsilon(x) \subset U$ and $B_\delta(x) \subset V$. Take $\eta = \min(\varepsilon, \delta)$. Then $B_\eta(x) \subset B_\varepsilon(x)$, $B_\eta(x) \subset B_\varepsilon(x)$; hence $B_\eta(x) \subset U \cap V$. Thus $U \cap V$ is open, and T2 is satisfied. Now T3. Let U_α be a family of open sets. If $x \in \cup U_\alpha$, then $x \in U_\alpha$ for some α. There is $B_\varepsilon(x) \subset U_\alpha$, hence $B_\varepsilon(x) \subset \cup U_\alpha$. Thus $\cup U_\alpha$ is open, and T3 satisfied. Hence the open sets in $\mathbb{R}^n$ defined as above satisfy the axioms of a topology T1, T2, T3.

Problem 5. T1 is satisfied because $\varnothing, X$ are subsets of X. T2, T3 are satisfied because the intersection of subsets is a subset and the union of subsets is a subset.

Problem 6. T1 is satisfied by the definition. The intersections and unions of $\varnothing, X$ are either $\varnothing$ or X, hence T2, T3 are satisfied.

Problem 7. $\varnothing$ is in $\mathcal{F}$ by the definition, and X is the complement of $\varnothing$, which is a subset with zero elements. Hence T1 is satisfied. To check T2 and T3 notice first that if $\varnothing$ is among the sets under consideration, the result is either $\varnothing$, hence open (for intersections), or does not change if we omit $\varnothing$ (for unions). So it suffices to check only the complements of finite sets. Notice now that the union of a finite family of finite subsets is finite, and the intersection of any collection of finite subsets is finite. Passing to the complements, we see that a finite intersection of the complements of finite subsets is the complement of a finite subset; also an arbitrary union of the complements of finite subsets is the complement of a finite subset. Hence T2 and T3 are satisfied. Hence $\mathcal{F}$ is a topology. In the case of a finite X, every subset is finite, hence the complements of finite subsets are all subsets, i.e., $\mathcal{F}$ coincides with the discrete topology. (We have used the following relations valid for any sets: $X \setminus (\cap A_\alpha) = \cup(X \setminus A_\alpha)$, $X \setminus (\cup A_\alpha) = \cap(X \setminus A_\alpha)$.)

Problem 8. A nonempty open set in $\mathbb{R}$ considered with the cofinite topology is the complement of a finite number of points and has the appearance $(-\infty, x_1) \cup (x_1, x_2) \cup \ldots \cup (x_{N-1}, x_N) \cup (x_N, \infty)$. Hence it is open in the usual sense. On the other hand, a finite interval (a, b), which is open in the usual sense, has an infinite complement, hence does not belong to the cofinite topology.

Problem 9. From Problem it follows that every open set in the sense of $\mathcal{F}$ is also open in the usual sense, but the converse is not true. Hence the map $(\mathbb{R}, \mathcal{E}) \to (\mathbb{R}, \mathcal{F})$, $x \mapsto x$, is continuous, while the map $(\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{E})$, $x \mapsto x$ is discontinuous. In particular, it is not a homeomorphism.

Problem 10. For any $\boldsymbol{x} \in \mathbb{R}^n$, we have $1 + |\boldsymbol{x}| > |\boldsymbol{x}|$. Hence

$$|f(\boldsymbol{x})| = \frac{|\boldsymbol{x}|}{1 + |\boldsymbol{x}|} < 1,$$

so $f(\boldsymbol{x})$ is indeed in B^n. Now, let us find the inverse map. If $\boldsymbol{y} = f(\boldsymbol{x})$, then $\boldsymbol{x} = \boldsymbol{y}(1 + |\boldsymbol{x}|)$. We have the equation $|\boldsymbol{x}| = \boldsymbol{y}(1 + |\boldsymbol{x}|)$ for $|\boldsymbol{x}|$, and from it $|\boldsymbol{x}| = \frac{|\boldsymbol{y}|}{1-|\boldsymbol{y}|}$ (which is well defined for $|\boldsymbol{y}| < 1$). Hence $1 + |\boldsymbol{x}| = \frac{1}{1-|\boldsymbol{y}|}$. Finally, the inverse map $B^n \to \mathbb{R}^n$ is given by

$$f^{-1}(\boldsymbol{y}) = \frac{\boldsymbol{y}}{1 - |\boldsymbol{y}|},$$

thus it is continuous. So we have a homeomorphism, as desired.

Problem 11. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$. We write $P = (\boldsymbol{x}, x_{n+1})$ for $P = (x_1, \ldots, x_{n+1})$. Likewise we denote by $\boldsymbol{u} = (u_1, \ldots, u_n)$ a point in $\mathbb{R}^n$. Let $S^n \subset \mathbb{R}^{n+1}$ be specified by the equation

$$x_1^2 + \ldots + x_n^2 + x_{n+1}^2 = 1.$$

We choose the point $N = (0, \ldots, 0, 1)$ as the center of the projection. There are two maps: $F\colon S^n \setminus \{N\} \to \mathbb{R}^n$ and $F^{-1}\colon \mathbb{R}^n \to S^n \setminus \{N\}$. If (NP) is the line through N and $P = (\boldsymbol{x}, x_{n+1}) \in S^n \setminus \{N\}$, and $P' = (\boldsymbol{u}, 0)$ stands for the intersection of (NP) with the $x_{n+1} = 0$ plane, then $F\colon P \mapsto P'$ and $F^{-1}\colon P' \mapsto P$. To find the explicit formulae, we write P_t as a point on the line (NP), so $P_t = tN + (1-t)P = ((1-t)\boldsymbol{x}, t + (1-t)x_{n+1})$. The last coordinates vanishes when $t = \frac{-x_{n+1}}{1-x_{n+1}}$, hence $1 - t = \frac{1}{1-x_{n+1}}$. We get for F

$$\boldsymbol{u} = \frac{\boldsymbol{x}}{1 - x_{n+1}}.$$

For the inverse map, we solve for $\boldsymbol{x}$ and x_{n+1} having in mind the relation $|\boldsymbol{x}|^2 + x_{n+1}^2 = 1$. We get $\boldsymbol{x} = \boldsymbol{u}(1 - x_{n+1})$, hence we arrive at a quadratic equation (denoting $1 - x_{n+1} = w$): $|\boldsymbol{u}|^2 w^2 + (1-w)^2 = 1$, or $(1+|\boldsymbol{x}|^2)w^2 - 2w = 0$, from where we get (since $w = 1 - x_{n+1} \neq 0$) $w = 2/(|\boldsymbol{u}|^2 + 1)$, and

$$\boldsymbol{x} = \frac{2\boldsymbol{u}}{|\boldsymbol{u}| + 1}$$

$$x_{n+1} = \frac{|\boldsymbol{u}|^2 - 1}{|\boldsymbol{u}|^2 + 1}$$

It follows that both F and F^{-1} are continuous, hence yield a homeomorphism between $S^n \setminus \{N\}$ and $\mathbb{R}^n$.

Problem 12. From the previous solution we obtain $F\colon (x_1, x_2, x_3) \mapsto w \in \mathbb{C}$ where

$$w = \frac{x_1 + ix_2}{1 - x_3}$$

and for the inverse map $F^{-1}\colon w \mapsto (x_1, x_2, x_3)$ we obtain

$$\begin{aligned} x_1 + ix_2 &= \frac{2w}{w\bar{w} + 1} \\ x_3 &= \frac{w\bar{w} - 1}{w\bar{w} + 1}\,. \end{aligned}$$