§3 Fundamental topological properties

Problem 1. S^n and $\mathbb{R}^n$ cannot be homeomorphic, because S^n is compact and $\mathbb{R}^n$ is non-compact.

Problem 11. The closed n-disk $D^n = \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| \leqslant 1\}$ is a closed set in $\mathbb{R}^n$, since its complement $\mathbb{R}^n \setminus D^n = \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| > 1\}$ is open (by definition: every point can be surrounded by a small open ball not intersecting D^n), and it is obviously bounded. Hence by the Heine–Borel theorem it is compact.

Problem 12. $\mathbb{Z}$ is non-compact. It is a discrete space, and a discrete space is compact if and only if it is finite. Since $\mathbb{Z}$ is infinite, it is non-compact. Alternatively, one can view $\mathbb{Z}$ as a subspace of $\mathbb{R}$. It is unbounded, hence non-compact.

Problem 13. We use the facts that S^n is compact for all n and that $I = [0, 1]$ is compact (both follow from the Heine–Borel theorem).

(a) $S^n \times I$ is compact as a product of compact spaces.

(b) $S^n \times S^m$ is compact as a product of compact spaces.

(c) $T^n = S^1 \times \ldots \times S^1$ is compact as a product of compact spaces.

(d) Follows from part (c): $T^n \times I$ is compact as a product of compact spaces.

(e) Notice that $\mathbb{R}P^n = S^n/\sim$, where $v \sim -v$, $v \in S^n \subset \mathbb{R}^n$. Therefore $\mathbb{R}P^n$ is compact as a continuous image (an identification space) of a compact space.

(f) Similar to part (e). $\mathbb{C}P^n$ can be considered as an identification space of $S^{2n+1} \subset \mathbb{C}^{n+1}$, under an equivalence relation $v \sim e^{i\alpha}v$, $v \in S^{2n+1}$, $\alpha \in \mathbb{R}$. Hence $\mathbb{C}P^n$ is compact as an identification space of a compact space.

Problem 14. In the following E stands for the identity matrix; A^T means the transpose of A; $\bar{A}$ means the complex conjugate of a matrix A (i.e., the complex conjugation applied to each matrix entry).

(a) $SO(n)$ is specified by the equations $AA^T = E$, $\det A = 1$ in the space $\operatorname{Mat}(n, \mathbb{R})$ of all $n \times n$ real matrices, which is just $\mathbb{R}^{n^2}$. Since the LHSs of these equations are continuous functions of matrix entries, $SO(n)$ is a closed set in $\operatorname{Mat}(n, \mathbb{R})$ as the preimage of a closed set under a continuous map. The matrix equation $AA^T = E$ implies $\operatorname{tr}(AA^T) = \operatorname{tr} E = n$, i.e., $\sum (A_{ij})^2 = n$. Hence $SO(n)$ is contained in the sphere of radius $\sqrt{n}$ with center at the origin in $\operatorname{Mat}(n, \mathbb{R})$, thus is bounded. By the Heine–Borel theorem, $SO(n)$ is compact as a closed bounded subspace of a Euclidean space.

(b) Same as above. $O(n)$ is specified by the equation $AA^T = E$ in $\operatorname{Mat}(n, \mathbb{R})$. Hence it is closed and (as above) bounded, because it is contained in the sphere of radius $\sqrt{n}$. Hence, by the Heine–Borel theorem, $O(n)$ is compact.

(c) Similar to the above. $U(n)$ is specified by the equation $A\bar{A}^T = E$ in $\mathrm{Mat}(n, \mathbb{C})$. It is a closed set as the preimage of a closed set (the one-point set $\{E\}$) under a continuous map (the map $A \mapsto A\bar{A}^T$). Now, the equation $A\bar{A}^T = E$ implies $\mathrm{tr}(A\bar{A}^T) = \mathrm{tr}\, E = n$, i.e., $\sum |A_{ij}|^2 = n$, which is the equation of a sphere of radius $\sqrt{n}$ in $\mathrm{Mat}(n, \mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{2n^2}$. Therefore $U(n)$ is bounded. By the Heine–Borel theorem, $U(n)$ is compact.

(d) Same as above. We have just one extra equation $\det A = 1$, with a continuous LHS.

Problem 15. Consider a matrix A which is diagonal with the diagonal entries $e^t, 1, \ldots, 1$, where $t \in \mathbb{R}$. It belongs to $GL(n)$. The distance between A and any chosen matrix can be made as large as we wish by a choice of t. For example, the distance between A and the identity matrix equals $|e^t - 1|$; the distance between A and the zero matrix equals e^t, for $t > 0$. Hence $GL(n)$ is not bounded, therefore is not compact. (We measure distances using the metric $d(A, B) = \max |A_{ij} - B_{ij}|$, which is equivalent to the Euclidean distance.)

Problem 16. To show that $SL(n)$ is non-compact we can slightly modify the previous example. Take the diagonal matrix with the diagonal entries $e^t, e^{-t}, 1, \ldots, 1$. It belongs to $SL(n)$ and is as far as we wish from any fixed matrix, e.g., the zero matrix or the identity matrix. Therefore $SL(n)$ is unbounded and non-compact. (We assume that $n > 1$; otherwise, we have $SL(1) = \{1\} \subset \mathbb{R}$, i.e., a one-point space, which is compact.)

Problem 17. See below a solution for Problem 19, where the complex case is considered.

Problem 18. Using the result of problem 17, $\mathbb{R}P^n$ is (homeomorphic to) a subspace of a Euclidean space, which is Hausdorff. Hence $\mathbb{R}P^n$ is Hausdorff as a subspace of a Hausdorff space.

Problem 19. Similar to Problem 17. $\mathbb{C}P^n$ is defined as the identification space of $\mathbb{C}^{n+1} \setminus \{0\}$ w.r.t. the equivalence relation $\boldsymbol{v} \sim \boldsymbol{w}$ iff $\boldsymbol{w} = k\boldsymbol{v}$, $k \neq 0$, $k \in \mathbb{C}$. Geometrically that means the space of lines through the origin in $\mathbb{C}^{n+1}$. The idea of an embedding into Euclidean space is associating a linear operator with each line. For example, one can use reflection operators. Any line $L \subset \mathbb{C}^{n+1}$ is uniquely defined by the reflection operator that takes an arbitrary vector $\boldsymbol{v} = \boldsymbol{v}_{\parallel} + \boldsymbol{v}_{\perp} \in \mathbb{C}^{n+1}$ where $\boldsymbol{v}_{\parallel} \in L$ and $\boldsymbol{v}_{\perp}$ is orthogonal to L, to the vector $R_L(\boldsymbol{v}) = \boldsymbol{v}_{\parallel} - \boldsymbol{v}_{\perp}$. ($L$ is recovered as the subspace consisting of all vectors $\boldsymbol{v}$ such that $R_L(\boldsymbol{v}) = \boldsymbol{v}$.) Explicitly, if $\boldsymbol{a} \neq 0$ spans L, then for any $\boldsymbol{v}$ we have $\boldsymbol{v}_{\parallel} = \frac{(\boldsymbol{v},\boldsymbol{a})}{(\boldsymbol{a},\boldsymbol{a})}\,\boldsymbol{a}$ and $\boldsymbol{v}_{\perp} = \boldsymbol{v} - \boldsymbol{v}_{\parallel}$. It follows that $R_L(\boldsymbol{v}) = 2\,\frac{(\boldsymbol{v},\boldsymbol{a})}{(\boldsymbol{a},\boldsymbol{a})}\,\boldsymbol{a} - \boldsymbol{v}$.

We have a map $\boldsymbol{a} \mapsto R_L$ from $\mathbb{C}^{n+1} \setminus \{0\}$ to linear operators on $\mathbb{C}^{n+1}$; if we introduce a basis and write it using matrices, it will be clear that this map is continuous. It factors through a map $L \mapsto R_L$ from $\mathbb{C}P^n$ to the space of linear operators on $\mathbb{C}^{n+1}$, which, therefore, is continuous (by a property of the identification topology). Now, $\mathbb{C}P^n$ is compact as the continuous image of the sphere S^{2n+1} and the space of linear operators is Hausdorff (as a vector space with Euclidean topology). Hence the map $L \mapsto R_L$ is a homeomorphism on its image, which gives a desired embedding of $\mathbb{C}P^n$ into a Euclidean space. Any subspace of a Euclidean space is Hausdorff, therefore $\mathbb{C}P^n$ will also be Hausdorff.

Problem 21. $\mathbb{R}$ is path-connected, because any two points $x, y \in \mathbb{R}$ can be joined by a segment $x_t = (1-t)x + ty$, $t \in [0,1]$. Path-connectedness implies connectedness. (This follows from the fact that a finite segment $[a,b]$ is connected.)

Problem 22. **(a)** $\mathbb{Z}$ is disconnected as a discrete space with more than one point (for example, $\mathbb{Z} = \{n \in \mathbb{Z} \mid n \leqslant 0\} \cup \{n \in \mathbb{Z} \mid n > 0\}$ is the union of two disjoint open sets) and hence is not path-connected; **(b)** $\mathbb{R}$ is connected (proved at the lectures) and path-connected (obviously: any two points can be joined by a segment); **(c)** $O(2)$ is disconnected (indeed, consider the continuous map $\det\colon O(2) \to \mathbb{R}$; its image consists of 1 and -1, which is a disconnected subspace of $\mathbb{R}$).

Problem 23. The Klein bottle is connected as an identification space of the unit square $I^2 \subset \mathbb{R}^2$, which is path-connected, hence connected ("the continuous image of a connected space").

Problem 24. The same as for Problem 23. (T^2 is also an identification space of the square.)

Problem 25. Suppose S^2 and T^2 are homeomorphic. Consider a closed curve without self-intersections in S^2, i.e., a subspace $C \subset S^2$ homeomorphic to a circle. Then $S^2 \setminus C$ is disconnected. (This is intuitively clear – draw a picture – and you can assume it without proof.) On the other hand, for the torus considered as the surface of revolution of a circle in the xz-plane about the z-axis in $\mathbb{R}^3$, consider the section by a vertical plane (passing through the z-axis). It is a circle $S^1 \subset T^2$; notice that $T^2 \setminus S^1$ is connected. A homeomorphism $f\colon T^2 \to S^2$ would homeomorphically map $S^1 \subset T^2$ to some $C \subset S^2$ and homeomorphically map $T^2 \setminus S^1$ on $S^2 \setminus C$. But the space $T^2 \setminus S^1$ is connected and the space $S^2 \setminus C$ is disconnected, which gives a contradiction. Thus no homeomorphism between T^2 and S^2 is possible.

Problem 26. Every element $U \in U(n)$ can be joined by a path with the identity matrix as follows. Let $U = gDg^{-1}$ where $D = \operatorname{diag}(e^{ix_1}, \dots, e^{ix_n})$;

consider a continuous path $t \mapsto U_t = gD_tg^{-1}$ where $D_t = \operatorname{diag}(e^{itx_1}, \dots, e^{itx_n})$, $t \in [0,1]$. Then $U_0 = E$ (the identity matrix), $U_1 = U$. Hence the topological group $U(n)$ is path-connected (moreover, connected).