§5 Triangulations and Euler characteristic

Problem 1. **(a)** A triangulation of $I = [0,1]$ consists of two vertices and one edge, hence $c_0 = 2$, $c_1 = 1$, and $\chi(I) = 2 - 1 = 1$.

(b) A triangulation of I^2 can be obtained by cutting the square into two triangles by a diagonal; hence it contains four vertices, five edges and two 2-simplices. Hence $\chi(I^2) = 4 - 5 + 2 = 1$.

(c) A triangulation of S^1 will contain k vertices and k edges (the smallest possible k is 3). Hence $\chi(S^1) = k - k = 0$.

(d) A triangulation of S^2 can be obtained by viewing the sphere as homeomorphic to the boundary of the tetrahedron. Hence for this triangulation $c_0 = 4$, $c_1 = 6$, $c_2 = 4$, and we obtain $\chi(S^2) = 4 - 6 + 4 = 2$. Alternatively, S^2 can be viewed as homeomorphic to the boundary of the cube. To get a triangulation we have to subdivide each face of the cube (a square) into two triangles, by diagonals. Hence for this triangulation $c_0 = 8$, $c_1 = 18$, $c_2 = 12$, and again we obtain $\chi(S^2) = 8 - 18 + 12 = 2$.

(e) This is a cylinder. We can view it as an identification space of a rectangle $ABCD$ (with two parallel vertical sides, AB and DC, identified). Let us first consider the following triangulation of the rectangle $ABCD$. On the horizontal side AD take points $P_0 = A$, P_1, P_2, and so on, so that $P_N = D$ (for some N), and similarly on the side BC take points $Q_0 = B$, Q_1, and so on, so that $Q_N = C$. Hence we subdivided our rectangle into smaller rectangles $P_kQ_kQ_{k+1}P_{k+1}$. Subdividing each of them into two triangles by a diagonal (e.g., by P_kQ_{k+1}), we get a triangulation of the 'large' rectangle $ABCD$. Now, a triangulation of the cylinder is obtained by identifying P_0 with P_N and Q_0 with Q_N. (The number N now cannot be too small, in order to satisfy the axioms of a simplicial complex, namely, that the intersection of two simplices can be only by a single common face.) Hence for this triangulation of the cylinder we have $c_0 = 2N$ (N vertices for each horizontal side AD and BC, taking into account the identifications $P_0 = P_N$ and $Q_0 = Q_N$), $c_1 = 4N$ ($2N$ horizontal edges, N diagonal edges, and N vertical edges, if we take into account the identification $P_0Q_0 = P_NQ_N$), and $c_2 = 2N$ ($2N$ triangles from N rectangles). It follows that $\chi(S^1 \times I) = 2N - 4N + 2N = 0$.

(f) The closed Möbius strip M can be viewed as an identification space of a rectangle, similarly to the cylinder. In the notation as above, M is obtained from $ABCD$ by the identification of AB with CD (two vertical sides are identified with opposite orientations). We can use a triangulation similar to the above. Namely, we first triangulate $ABCD$ as above and then identify P_0 with Q_N and Q_0 with P_N (instead of the identifications $P_0 = P_N$ and $Q_0 = Q_N$ used above). This is the only difference, which does not affect the counting of vertices, edges and triangles. We obtain the same $c_0 = 2N$, $c_1 = 4N$, and $c_2 = 2N$, hence $\chi(M) = 0$.

Problem 2. **(a)** For I^2 we can consider a triangulation obtained by subdividing into two triangles by a diagonal, see the solution to Problem 1(b). We get $c_0 = 4$, $c_1 = 5$, $c_2 = 2$, giving $\chi(I^2) = 4 - 5 + 2 = 1$. Alternatively, we can consider subdividing I^2 into four smaller squares with side $1/2$ and then subdividing each of them by a diagonal. Then $c_0 = 9$, $c_1 = 16$, $c_2 = 8$, giving $\chi(I^2) = 9 - 16 + 8 = 1$. There are (infinitely) many other options as well.

(b) All triangulations of S^1 are described in the solution to Problem 1(c).

(c) See the solution to Problem 1(d). There are infinitely many other options.

Problem 3. **(a)** $4 - 4 + 1 = 1 = \chi(I^2)$.

(b) From any such tiling one can get a triangulation by subdividing each square by a diagonal. Hence the number of vertices will remain the same, the number of edges will increase by the number of squares, and the number of triangles (2-simplices) will be the number of squares doubled. Denote the number of vertices in the tiling by q_0, the number of edges by q_1, the number of squares by q_2. Hence $\chi = c_0 - c_1 + c_2 = q_0 - (q_1 + q_2) + 2q_2 = q_0 - q_1 + q_2$, as stated.

(c) $\chi(S^2) = q_0 - q_1 + q_2 = 8 - 12 + 6 = 2$

Problem 4. We first triangulate the unit square by I^2 subdividing it into 9 squares with side $1/3$ by lines parallel to the sides. Each square can be then subdivided into two triangles. It is convenient to denote the vertices of the triangulation by P_{ij}, where $i, j = 0, 1, 2, 3$. Here P_{ij} has coordinates $(\frac{i}{3}, \frac{j}{3})$. We assume that the coordinate lines go along the sides and the original vertices are $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. (Draw a picture!) For each of the surfaces a triangulation can be obtained by taking into account the corresponding identifications of I^2. The Euler characteristic can be calculated from these triangulations, or (using the result of Problem 3) directly from the tilings by small squares. It is easier not to count all the numbers involved explicitly, but only check the changes due to gluing.

(a) For the Klein bottle K we have to identify the side $P_{00}P_{30}$ with the side $P_{03}P_{33}$ (preserving orientation) and the side $P_{00}P_{03}$ with $P_{33}P_{30}$ (with opposite orientations). Hence we get the following identifications: $P_{00} = P_{03} = P_{30} = P_{33}$, $P_{01} = P_{32}$, $P_{02} = P_{31}$, $P_{10} = P_{13}$, $P_{20} = P_{23}$ (use the picture!). It follows that when we glue the Klein bottle from I^2, the number of vertices decreases by 7. The number of edges decreases by 6, since we identify the three edges on a horizontal side with the three edges on the parallel side, and the same for vertical sides. The number of triangles does not change. Counting the effect for the Euler characteristic, we conclude that $\chi(K) = \chi(I^2) - (7 - 6 + 0) = 1 - 1 = 0$.

(b) For $\mathbb{R}P^2$ we have to identify the antipodal points of the boundary of I^2 (those symmetric w.r.t. the center of the square). Hence the side $P_{00}P_{30}$ is identified with the side $P_{33}P_{03}$ and the side $P_{00}P_{03}$ with $P_{33}P_{30}$ (in both cases with opposite orientations). We get the following identifications for vertices: $P_{00} = P_{03}$, $P_{10} = P_{23}$, $P_{20} = P_{13}$, $P_{30} = P_{03}$, $P_{01} = P_{32}$, $P_{02} = P_{31}$. It follows that, compared with the triangulation of I^2, the number of vertices decreases by 6 (notice the difference with the Klein bottle!). In the same way as for the Klein bottle, the number of edges decreases by 6 (the different way of identifying edges does not affect the counting), and the number of triangles does not change. Altogether we have $\chi(\mathbb{R}P^2) = \chi(I^2) - (6-6+0) = 1-0 = 1$.

(c) The case of T^2 is very similar to that of the Klein bottle. We have to identify $P_{00}P_{30}$ with $P_{03}P_{33}$ and $P_{00}P_{03}$ with $P_{30}P_{33}$ (preserving orientations). Hence for vertices we have the identifications $P_{00} = P_{03} = P_{33} = P_{30}$, $P_{01} = P_{31}$, $P_{02} = P_{32}$, $P_{10} = P_{13}$, $P_{20} = P_{23}$. The number of vertices decreases by 7, the number of edges decreases by 6, and the number of triangles does not change, compared with the triangulation of I^2. It follows that $\chi(T^2) = \chi(I^2) - (7-6+0) = 1-1 = 0$.

Problem 5. We can apply the excision formula as follows. Let S^2_k stand for the sphere with k holes, i.e. S^2 with k disjoint open disks removed. Then

$$S^2 = S^2_k \cup \{k \text{ closed disks}\},$$

and the intersection of S^2_k with these closed disks consists of k (disjoint) circles. Hence, by the excision formula, we have the following equation: $\chi(S^2) = \chi(S^2_k) + k\chi(D^2) - k\chi(S^1)$. Here D^2 stands for the closed disk, which is homeomorphic to the square. We know that $\chi(S^2) = 2$, $\chi(D^2) = \chi(I^2) = 1$ and $\chi(S^1) = 0$. Hence we have: $2 = \chi(S^2_k) + k$, or $\chi(S^2_k) = 2 - k$.

Alternatively, if we consider a sufficiently fine triangulation of S^2, cutting a hole amounts to the removing of a 2-simplex (the boundary of the simplex remains), and cutting k holes amounts to the removing of k 2-simplices (the boundaries of them remaining). Hence the number of vertices and the number of edges do not change under such operation, while the number of 2-simplices decreases by k. Hence $\chi = c_0 - c_1 + c_2$ also decreases by k, and we obtain $S^2_k = S^2 - k = 2 - k$.

Problem 6. **(a)** The surface H^2_g is obtained from the sphere S^2 by cutting $2g$ holes and gluing in g handles (homeomorphic to the cylinder $S^1 \times I$). We can apply the excision formula to obtain the equation $\chi(H^2_g) = \chi(S^2_{2g}) + g\,\chi(S^1 \times I) - 2g\,\chi(S^1)$, where we denoted by S^2_k the sphere with k holes. Either by a suitable triangulation or applying the excision formula again (see Problem 5), we can obtain $\chi(S^2_k) = \chi(S^2) - k = 2 - k$. Also, by triangulations, we have $\chi(S^1) = \chi(S^1 \times I) = 0$ (see Problem 1(e) and (f)). Hence, we obtain $\chi(H^2_g) = \chi(S^2_{2g}) = 2 - 2g$.

(b) Similar to part (a). The surface M^2_ν is obtained from the sphere S^2 by cutting ν holes and gluing in ν closed Möbius strips. Hence, by the excision formula, $\chi(M^2_\nu) = \chi(S^2_\nu) + \nu\,\chi(M) - \nu\,\chi(S^1) = 2 - \nu$ (where we denoted by M the Möbius strip).

Problem 7. Similar to Problem 6. The surface $X_{g,\nu}$ is obtained from the sphere by cutting $2g+\nu$ holes and gluing in g cylinders (along $2g$ boundary circles) and ν closed Möbius strips (along the remaining ν boundary circles). Hence, by the excision formula we have $\chi(X_{g,\nu}) = \chi(S^2_{2g+\nu}) + g\,\chi(S^1 \times I) + \nu\,\chi(M) - (2g+\nu)\,\chi(S^1) = 2 - 2g - \nu$.

Problem 8. The surface $X_{g,\nu}$ is non-orientable because it contains a Möbius strip (we assume that $\nu > 0$). Hence, by the classification theorem for closed surfaces, it is homeomorphic to the standard non-orientable surface M^2_k for some k. To find k, consider the Euler characteristic. Since $\chi(M^2_k) = 2 - k$ and $\chi(X_{g,\nu}) = 2 - 2g - \nu$ (using the result of Problem 7), we must have $k = 2g + \nu$. Hence $X_{g,\nu} \cong M^2_{2g+\nu}$, as claimed.