§2 Topological constructions

Problem 1. Consider a map $f\colon \mathbb{R} \to \mathbb{R}_+$, $f(x) = e^x$. It is continuous. The inverse map, $f^{-1}\colon \mathbb{R}_+ \to \mathbb{R}$, $f^{-1}(y) = \ln y$, is also continuous. Hence they together give a desired homeomorphism $\mathbb{R} \cong \mathbb{R}_+$.

Problem 2. Denote this subspace by H. We can solve for z:

$$z = \sqrt{x^2 + y^2 + 1}.$$

The map $f\colon H \to \mathbb{R}^2$, $(x, y, z) \mapsto (x, y)$ is continuous and the inverse map $(x, y) \mapsto (x, y, \sqrt{x^2 + y^2 + 1})$ is also continuous. Hence they define a homeomorphism between H and $\mathbb{R}^2$.

Problem 3. The space $SU(2)$ as a subspace of the space of all complex 2×2 matrices is specified by the equations $A\bar{A}^T = E$ and $\det A = 1$. Here bar over a matrix means complex conjugation of its entries, A^T means the transpose of A. Writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

gives the following set of equations in terms of the matrix entries a, b, c, d:

$$\begin{aligned} a\bar{a} + b\bar{b} &= 1 \\ a\bar{c} + b\bar{d} &= 0 \\ c\bar{c} + d\bar{d} &= 1 \\ ad - bc &= 1 \end{aligned}$$

The second equation means that $(c, d) = u(-\bar{b}, \bar{a})$ where u is an arbitrary factor. Taking into account the third equation leads to $u\bar{u} = 1$, i.e., $u = e^{it}$, $t \in \mathbb{R}$. Hence, without the condition $\det A = 1$, i.e., for $U(2)$, we get an explicit description

$$A = \begin{pmatrix} a & b \\ -e^{it}\bar{b} & e^{it}\bar{a} \end{pmatrix}$$

where $a, b \in \mathbb{C}$ are subject to the condition $a\bar{a} + b\bar{b} = 1$ and $t \in \mathbb{R}$. Imposing $\det A = 1$ gives $e^{it} = 1$. Hence, finally, for $A \in SU(2)$

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

where $a\bar{a} + b\bar{b} = 1$, and we can identify such matrices with unit vectors in $\mathbb{C}^2 = \mathbb{R}^4$, so $SU(2) \cong S^3$.

Problem 9. Suppose f is continuous. Then both compositions $f_1 = p_1 \circ f\colon Z \to X$ and $f_2 = p_2 \circ f\colon Z \to Y$ are continuous, since the canonical projections p_1 and p_2 are continuous. Conversely, suppose the compositions $f_1 = p_1 \circ f\colon Z \to X$ and $f_2 = p_2 \circ f\colon Z \to Y$ are continuous. That means that for arbitrary open sets $U \in \mathcal{O}_X$, $V \in \mathcal{O}_Y$ the sets $f_1^{-1}(U) = (p_1 \circ f)^{-1}(U) = f^{-1}(U \times Y)$ and $f_2^{-1}(V) = (p_2 \circ f)^{-1}(V) = f^{-1}(X \times V)$ are open in Z. Take an open set in $X \times Y$. It suffices to consider a set of the form $U \times V$ where $U \in \mathcal{O}_X$, $V \in \mathcal{O}_Y$. As $U \times V = (U \times Y) \cap (X \times V)$, we have $f^{-1}(U \times V) = f^{-1}(U \times Y) \cap f^{-1}(X \times V)$, hence open.