§1 Topological spaces and continuous maps

Problem 1. List all possible topologies on a two-point set: $X = \{a, b\}$.

Problem 2. Prove that the open sets in $\mathbb{R}^n$ (in the usual sense) satisfy T1, T2, T3, thus making a topology.

Problem 3. Prove that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ define the same topology (referred to as the "usual", or "Euclidean", topology). Here

$$
\begin{aligned}
d_1(a,b) &= \sum_{k=1}^{n} |a_k - b_k| \\
d_2(a,b) &= \left(\sum_{k=1}^{n} (a_k - b_k)^2 \right)^{1/2} \\
d_\infty(a,b) &= \max_{k=1\dots n} |a_k - b_k| \,.
\end{aligned}
$$

For this do the following.
(a) For $n = 2$, sketch an ε-neighborhood of an arbitrary point $a \in \mathbb{R}^2$ w.r.t. each of the metrics d_1, d_2 and d_∞. (Without loss of generality you may consider $a = 0$.)
(b) Show that an ε-neighborhood of an arbitrary point $a \in \mathbb{R}^2$, in the sense of each of the metrics d_1, d_2 and d_∞, contains ε'-neighborhoods in the sense of each of the two other metrics, for appropriate ε'. (Use pictures.)
(c) For arbitrary n prove that inequalities of the form

$$d_i(a,b) \leqslant C \cdot d_j(a,b)$$

with some positive constants C hold for each pair i, j, where $i, j = 1, 2, \infty$. Constants are different for different pairs i, j. (What happens with these constants when $n \to \infty$, and which of the inequalities survive in the limit?)
(d) Deduce from **(c)** that the statement of **(b)** is true for any finite n.
(e) Deduce from here the main statement, i.e., that the different metrics d_1, d_2 and d_∞ give the same topology on $\mathbb{R}^n$.

Problem 4. Let S^n denote the unit sphere with center at the origin in $\mathbb{R}^{n+1}$. One can introduce the notion of distance between points of S^n in two ways: either as the Euclidean distance or as the "distance measured along the sphere", i.e., the length of the shortest arc of the great circle joining these points. ("Great circles" are sections of the sphere by two-dimensional planes through the center.)
(a) Show that these two distances define the same topology on S^n (referred to as the "standard" topology of the sphere).
(b) Show that open sets in this topology are precisely the intersections of the open sets in $\mathbb{R}^{n+1}$ with S^n.

Problem 5. Check that the *discrete topology* (i.e., the collection of all subsets) is indeed a topology for an arbitrary set X.

Problem 6. Check that the collection $\{\varnothing, X\}$ for an arbitrary set X (called the *indiscrete topology*) is indeed a topology.

Problem 7. For an arbitrary set X consider the collection $\mathcal{F}$ consisting of the empty set $\varnothing$ and the complements of all finite subsets (i.e., $X \setminus A$ where A is finite). Prove that $\mathcal{F}$ is a topology, called the *cofinite topology* on X. What is $\mathcal{F}$ if X is finite?

Problem 8. Sketch a nonempty open set in $\mathbb{R}$ with the cofinite topology. Is it open in the usual sense? Does every subset in $\mathbb{R}$ open in the usual topology belong to the cofinite topology? (If the answer is negative, give a counterexample).

Problem 9. Let $\mathcal{F}$ denote the cofinite topology on $\mathbb{R}$ and $\mathcal{E}$, the usual (Euclidean) topology. Is the "identity" map $f\colon (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{E})$, $f\colon x \mapsto x$, a homeomorphism? *Hint:* check whether both f and f^{-1} are continuous.

Problem 10. Let $B^n \subset \mathbb{R}^n$ denote the open unit ball in $\mathbb{R}^n$ with center at the origin, i.e., $B^n = \{\boldsymbol{x} \in \mathbb{R}^n \mid |\boldsymbol{x}| < 1\}$. Prove that the map $f\colon \mathbb{R}^n \to B^n$,

$$f\colon \boldsymbol{x} \mapsto \frac{\boldsymbol{x}}{1 + |\boldsymbol{x}|} \in B^n$$

is well-defined (i.e., that its image is indeed in B^n) and gives a homeomorphism $\mathbb{R}^n \cong B^n$.

Problem 11. Let S^n denote the unit sphere with center at the origin in $\mathbb{R}^{n+1}$. Consider $\mathbb{R}^n$ as the plane of the first n coordinates in $\mathbb{R}^{n+1}$. Let N denote the "north pole" of the sphere, i.e., $N = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$. Define the *stereographic projection* as the map from $S^n \setminus \{N\}$ to $\mathbb{R}^n$ sending a point P of the sphere to the point P' of $\mathbb{R}^n$ which is the intersection with the straight line through N and P. Denote this map by F.
(a) Denoting the coordinates of $P \in S^n$ by $(x_1, \ldots, x_{n+1})$ and the coordinates of $P' \in \mathbb{R}^n$ by $(u_1, \ldots, u_n)$, give explicit formulas for F.
(b) Find the explicit formulas for the inverse map F^{-1}.
(c) Show that the stereographic projection establishes a homeomorphism between the topological spaces $S^n \setminus \{\mathit{point}\}$ and $\mathbb{R}^n$.

Problem 12. Identifying $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, write the formulas for the stereographic projection and its inverse as maps $S^2 \setminus \{N\} \to \mathbb{C}$ and $\mathbb{C} \to S^2 \setminus \{N\}$. (Denote $u_1 + iu_2 = w \in \mathbb{C}$.)