§3 Fundamental topological properties

Problem 1. Show that the following sets are closed:
(a) $I = [0, 1] \subset \mathbb{R}$, (b) $I^n \subset \mathbb{R}^n$, (c) $S^n \subset \mathbb{R}^{n+1}$, (d) the set of all non-invertible matrices in $\mathrm{Mat}(n)$, (e) the set of all orthogonal matrices in $\mathrm{Mat}(n)$, (f) the set of all unitary matrices in $\mathrm{Mat}(n, \mathbb{C})$.

Problem 2. Show that the following spaces are Hausdorff:
(a) I^n, (b) S^n, (c) T^2, (d) T^n.

Problem 3. Show that the Klein bottle is Hausdorff. *Hint:* investigate various "types" of points on I^2 (on the boundary, inside, etc.).

Problem 4. Show that an indiscrete space is non-Hausdorff.

Problem 5. Show that $\mathbb{R}$ with the cofinite topology is non-Hausdorff.

Problem 6. Prove that if both X and Y are Hausdorff then the product space $X \times Y$ is Hausdorff.

Problem 7. Consider a topological space X. Suppose $\mathcal{B}$ is a base of the topology of X. Show that if every cover of X by elements of $\mathcal{B}$ contains a finite subcover then X is compact.

Problem 8. Show that if two spaces X and Y are homeomorphic and X is compact then Y is also compact.

Problem 9. Prove that S^n is compact for all n.

Problem 10. Can S^n and $\mathbb{R}^n$ be homeomorphic? Justify your answer.

Problem 11. Prove that the closed n-disk $D^n = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, |\boldsymbol{x}| \leqslant 1\}$ is compact for all n.

Problem 12. Is $\mathbb{Z}$ compact? Justify your answer.

Problem 13. Show that the following spaces are compact:
(a) $S^n \times I$; **(b)** $S^n \times S^m$; **(c)** T^n; **(d)** $T^n \times I$; **(e)** $\mathbb{R}P^n$; **(f)** $\mathbb{C}P^n$.

Problem 14. Prove that the following topological groups are compact:
(a) $SO(n)$; **(b)** $O(n)$; **(c)** $U(n)$; **(d)** $SU(n)$.

Problem 15. Show that the topological group $GL(n)$ is non-compact. *Hint:* give an example of a matrix $g \in GL(n)$ which is as far as you wish from the zero matrix (look among the diagonal matrices).

Problem 16. Show that the topological group $SL(n)$ is non-compact. *Hint:* use the example from the previous problem modifying it if necessary.

Problem 17. Consider the straight line in $\mathbb{R}^{n+1}$ (through the origin) spanned by a nonzero vector $\boldsymbol{v} = (x^1, \ldots, x^{n+1})$. Let $R_{\boldsymbol{v}}$ denote the reflection of the space $\mathbb{R}^{n+1}$ in the plane orthogonal to $\boldsymbol{v}$ (i.e., the map which fixes all the vectors in this plane and sends $k\boldsymbol{v}$ to $-k\boldsymbol{v}$).
(a) Write down explicitly $R_{\boldsymbol{v}}(\boldsymbol{y})$ for an arbitrary vector $\boldsymbol{y} \in \mathbb{R}^{n+1}$. *Hint:* decompose $\boldsymbol{y}$ into the sum of a vector proportional to $\boldsymbol{x}$ and a vector in the orthogonal plane.
(b) Write the operator $R_{\boldsymbol{v}}$ as a matrix.
(c) Check that $[\boldsymbol{v}] \mapsto R_{\boldsymbol{v}}$ is a well-defined map $\mathbb{R}P^n \to \mathrm{Mat}(n+1)$. Check that it is injective.
(d) Show that the above map is continuous (use the properties of identification topology).
(e) Using the homeomorphism theorem, show that the map above is a homeomorphism of $\mathbb{R}P^n$ onto a subspace of $\mathrm{Mat}(n+1) \cong \mathbb{R}^{(n+1)^2}$.

Problem 18. Show that $\mathbb{R}P^n$ is Hausdorff.

Problem 19. Construct an embedding of $\mathbb{C}P^n$ into a Euclidean space. Show that $\mathbb{C}P^n$ is Hausdorff.

Problem 20. Suppose that topological spaces X and Y are homeomorphic.
(a) Prove that X is connected if and only if Y is connected.
(b) Prove that X is path-connected if and only if Y is path-connected.

Problem 21. Assuming without proof that every finite segment $[a, b]$ is connected, show that $\mathbb{R}$ is connected.

Problem 22. Are the following spaces connected? Path-connected?
(a) $\mathbb{Z}$; **(b)** $\mathbb{R}$; **(c)** $O(2)$.

Problem 23. Show that the Klein bottle is connected.

Problem 24. Show that the torus T^n is connected.

Problem 25. Using the notion of connectedness give an argument showing that S^2 and T^2 are not homeomorphic.

Problem 26. Use the fact that for every matrix $U \in U(n)$ there is an invertible matrix g such that $U = gDg^{-1}$ where $D = \mathrm{diag}(e^{ix_1}, \ldots, e^{ix_n})$ (here $x_1, \ldots, x_n$ are real numbers) to show that the topological group $U(n)$ is connected. *Hint:* show that every point of $U(n)$ can be joined by a path with the identity matrix.

Problem 27. Use the fact that every matrix $g \in GL(n, \mathbb{C})$ can be presented as a product $g = RU$ where R is a Hermitian matrix with positive eigenvalues and U is a unitary matrix to show that $GL(n, \mathbb{C})$ is connected. *Hint:* use the results of the previous problem; consider the matrix logarithm to check that the space of Hermitian matrices with positive eigenvalues is path-connected.

Problem 28. **(a)** Show that $SO(2)$ is connected.
(b) Show that $SO(3)$ is connected.
(c) Show that $SO(n)$ is connected for all n.
Hint: for $n \geqslant 3$ use an argument similar to that in Problem 26.

Problem 29. A *path-connected component* of a topological space consists of all points that can be joined by a path (i.e., two points belong to the same component if they can be joined by a path, and to different components otherwise). The path-connected component of a point is the path-connected component containing this point.
(a) Prove that $O(n)$ has exactly two path-connected components.
(a) Prove that $GL(n)$ has exactly two path-connected components. *Hint:* use an argument similar to that in Problem 27.

Problem 30. A *component* (or *connected component*) of a topological space X is a maximal connected subset, i.e., such a connected subset that there is no "larger" connected subset containing this one.
(a) Show that each component is a closed set.
(b) Show that X is the union of disjoined components.
(c) Suppose X has only a finite number of distinct components. Show that each component is an open set.

Problem 31. Suppose G is a topological group.
(a) Show that the component of identity in G is a normal subgroup.
(a) Show that the path-connected component of identity in G is a normal subgroup.
(Remark. Connected components and path-connected components may not coincide. However, for locally-Euclidean spaces, for which connectedness and path-connectedness are equivalent, components and path-connected components are the same.)