§2 Topological constructions

Problem 1. Define $\mathbb{R}_+$ as a subspace of $\mathbb{R}$ consisting of all positive numbers. Show that $\mathbb{R}$ and $\mathbb{R}_+$ are homeomorphic. *Hint:* use the exponential.

Problem 2. Prove that the subspace of $\mathbb{R}^3$ specified by the conditions $x^2 + y^2 - z^2 = -1$, $z > 0$, is homeomorphic to $\mathbb{R}^2$. *Hint:* make a sketch; you might use a projection to establish a required homeomorphism.

Problem 3. Prove that the space $SU(2)$ consisting of unitary 2×2-matrices satisfying $\det A = 1$ is homeomorphic to S^3. *Hint:* write down the equations specifying $SU(2)$ as a subspace of all complex 2×2-matrices explicitly.

Problem 4. Prove that for an identification space X/R, a map $f\colon\ X/R \to Y$ is continuous if and only if $f \circ p\colon\ X \to Y$ is continuous.

Problem 5. Let R be the following relation on $\mathbb{R}$: xRy if and only if $x-y \in \mathbb{Z}$. Check that it is an equivalence relation.

Problem 6. The identification space of $\mathbb{R}$ w.r.t. the equivalence relation defined in the previous problem is denoted $\mathbb{R}/\mathbb{Z}$. Let $p\colon\ \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the canonical projection. Show that p is an *open map* (i.e., the image of every open set is open). *Hint:* consider a base of topology on $\mathbb{R}$ consisting of intervals of length $< 1/2$ and show that $p^{-1}(p(U))$ is open for any such interval U.

Problem 7. Check that the sets of the form $p((a,b))$ where $|a-b| < 1/2$ make a base of the identification topology for $\mathbb{R}/\mathbb{Z}$.

Problem 8. Consider a map $f\colon\ \mathbb{R}/\mathbb{Z} \to S^1$, where $f\colon\ [x] \mapsto e^{2\pi i x}$. (The circle is considered as a subspace of $\mathbb{C}$: $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$.)
(a) Check that f is well defined.
(b) Prove that f is continuous. *Hint:* use Problem 4.
(c) Check that $g\colon\ S^1 \to \mathbb{R}/\mathbb{Z}$, $g\colon\ z \mapsto [\frac{\arg z}{2\pi}]$ is the inverse map for f.
(d) Show that g is continuous (hence f and g establish a homeomorphism $\mathbb{R}/\mathbb{Z} \cong S^1$). *Hint:* use the base constructed in Problem 7.

Problem 9. Prove that a map $f\colon\ Z \to X \times Y$ is continuous if and only if the maps $f_1 = p_1 \circ f\colon\ Z \to X$ and $f_2 = p_2 \circ f\colon\ Z \to Y$ are continuous.

Problem 10. The real projective space $\mathbb{R}P^n$ is defined as the identification space of $\mathbb{R}^{n+1} \setminus \{0\}$ w.r.t. the following equivalence relation: $\boldsymbol{v} \sim \boldsymbol{u}$ if and only if $\boldsymbol{v} = a\boldsymbol{u}$, $a \neq 0$ (a nonzero real number). Check that $\mathbb{R}P^n$ is homeomorphic to an identification space of S^n. What is the corresponding equivalence relation on the sphere? *Hint:* use unit vectors.

Problem 11. The complex projective space $\mathbb{C}P^n$ is defined similarly as the identification space of $\mathbb{C}^{n+1} \setminus \{0\}$ w.r.t. the equivalence relation $\boldsymbol{v} \sim \boldsymbol{u}$ if and only if $\boldsymbol{v} = a\boldsymbol{u}$, $a \neq 0$ (a nonzero complex number). Check that $\mathbb{C}P^n$ is homeomorphic to an identification space of S^{2n+1}. What is the corresponding equivalence relation on the sphere? *Hint:* use unit vectors w.r.t. the Hermitian scalar product $(\boldsymbol{u}, \boldsymbol{v}) = \sum u^k \bar{v}^k$.

Problem 12. Show that $\mathbb{R}P^1 \cong S^1$.

Problem 13. Show that $\mathbb{C}P^1 \cong S^2$.

Problem 14. Show that $\mathbb{R}P^n = \mathbb{R}^n \cup \mathbb{R}P^{n-1}$. More precisely, show that in $\mathbb{R}P^n$ there are subspaces homeomorphic to $\mathbb{R}^n$ and to $\mathbb{R}P^{n-1}$, and that the whole space is their union. (This gives an inductive description of the structure of the projective space.) *Hint:* use coordinates in $\mathbb{R}^{n+1}$ to specify subspaces in $\mathbb{R}P^n$.

Problem 15. Formulate and prove a similar statement for $\mathbb{C}P^n$.

Problem 16. Show that $\mathbb{R}^2 \setminus \{(0,0)\}$ is homeomorphic to the product space $\mathbb{R} \times S^1$ (infinite cylinder). *Hint:* use polar coordinates.

Problem 17. Show that the 2-torus defined as a surface of revolution in $\mathbb{R}^3$ (i.e., as a subspace of $\mathbb{R}^3$) is homeomorphic to each of the following spaces:
(a) the identification space of $\mathbb{R}^2$ w.r.t. the equivalence relation: $(x, y) \sim (x+1, y)$ and $(x, y) \sim (x, y+1)$;
(b) the identification space of I^2 (the unit square) w.r.t. the equivalence relation: $(0, y) \sim (1, y)$, $(x, 0) \sim (x, 1)$, and the points inside the square are equivalent only to themselves;
(c) the product space $S^1 \times S^1$.

Problem 18. Show that the topological group $U(n)$ is homeomorphic to the product space $S^1 \times SU(n)$, where $SU(n)$ is defined as the subspace of $U(n)$ consisting of all matrices with unit determinant.