§1 Topological spaces and continuous maps

Our **main goal** in this section is to rigorously define what is meant by a "topological equivalence" and to specify a natural class of objects to which such notion will be applicable.

The first step is to say that two objects, say, X and Y are topologically equivalent if one can be obtained from another "continuously", "without ruptures or gluing". Developing it further, we say that it means that X can be "continuously mapped" to Y, and vice versa, with the two maps $f\colon X \to Y$ and $g\colon Y \to X$ being mutually inverse.

It follows then that our task becomes more specific: we have to define continuous maps and identify objects for which such maps will make sense. To achieve this, we shall make a digression to familiar notions, partly following the historical development of the subject.

Recall that for a function $f\colon \mathbb{R} \to \mathbb{R}$ (one can consider functions defined on an interval $(\alpha, \beta) \subset \mathbb{R}$ instead, but it is not essential for our purposes), to be *continuous at a point* x_0 means that for every $\varepsilon > 0$ there is $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all x satisfying $|x - x_0| < \delta$. In words: the values $f(x)$ can be made arbitrarily close to the value $f(x_0)$ by taking x sufficiently close to x_0.

This can reformulated more geometrically, by using the language of "ε-neighborhoods". For an arbitrary number $\varepsilon > 0$ by the *ε-neighborhood of a point* $x_0 \in \mathbb{R}$ is meant the set of all $x \in \mathbb{R}$ such that the distance between x and x_0 is less than ε. Recall that the distance $d(a, b)$ between two points $a, b \in \mathbb{R}$ is simply $|a - b|$, so the ε-neighborhood of $a \in \mathbb{R}$ is the interval $(a - \varepsilon, a + \varepsilon)$. The continuity of the function f at x_0 is the condition that *for any ε-neighborhood $V_\varepsilon(y_0)$ of $y_0 = f(x_0)$ there exists a δ-neighborhood $U_\delta(x_0)$ of x_0 such that $f(U_\delta(x_0)) \subset V_\varepsilon(y_0)$.*

In this form the notion of continuity immediately carries over to $\mathbb{R}^n$ and to abstract "metric spaces".

Definition 1.1. A *metric space* is a set X with a function $d\colon X \times X \to \mathbb{R}$ called the *metric* or the *distance function*, whose value $d(a, b)$ for any two points $a, b \in X$ is called the *distance* between a and b, satisfying the following properties:

(M1) $d(a, b) \geqslant 0$ and $d(a, b) = 0$ if and only if $a = b$

(M2) $d(a, b) = d(b, a)$

(M3) $d(a, b) \leqslant d(a, c) + d(c, b)$ (*triangle inequality*).

For a point a in a metric space X, the *ball* (or *open ball*) with center a and radius R, shortly: the R-ball around a, is defined as the set

$$B_R(a) = \{x \in X \mid d(x, a) < R\}.$$

The *ε-neighborhood* of a point a is the same as the ε-ball around a.

Definition 1.2. A map of metric spaces $f\colon X \to Y$ is *continuous at* $x_0 \in X$ if for every ε-neighborhood $V_\varepsilon(y_0) = B_\varepsilon(y_0) \subset Y$ of $y_0 = f(x_0) \in Y$ there is a δ-neighborhood $U_\delta(x_0) = B_\delta(x_0) \subset X$ of x_0 such that $f(U_\delta(x_0)) \subset V_\varepsilon(y_0)$.

Example 1.1. For $\mathbb{R}$ with the metric $d(a,b) = |a-b|$, the 'ball' $B_\varepsilon(a)$ is just the open interval $(a-\varepsilon, a+\varepsilon)$.

We come back to the usual notion of continuity for functions $\mathbb{R} \to \mathbb{R}$.

Example 1.2. There are several natural ways to introduce metric into $\mathbb{R}^n$. One can check that the following functions all satisfy the above conditions M1, M2 and M3:

$$\begin{aligned} d_1(a,b) &= \sum_{k=1}^{n} |a_k - b_k| \\ d_2(a,b) &= \left(\sum_{k=1}^{n} (a_k - b_k)^2 \right)^{1/2} \\ d_\infty(a,b) &= \max_{k=1\dots n} |a_k - b_k| \,. \end{aligned}$$

Exercise: sketch open 1-balls around the origin $O \in \mathbb{R}^n$ for $n = 1,2,3$. You will see that d_1, d_2 and d_∞ agree for $n=1$ but are different for $n>1$.

Claim: for finite dimension n, the different metrics d_1, d_2 and d_∞ lead to the same notion of continuity for functions $\mathbb{R}^n \to \mathbb{R}$ (the same holds for maps $\mathbb{R}^n \to \mathbb{R}^m$ for finite n and m).

Indeed, in $\mathbb{R}^n$ one can always 'inscribe' a 'ball' in the sense of one metric into a 'ball' in the sense of another metric, for the metrics d_1, d_2 and d_∞, and in this way translate the condition of continuity formulates in terms of one type of 'balls' into the same condition in terms of another type of 'balls'. (Make a sketch in $\mathbb{R}^2$!) Thus the two conditions must be equivalent.

The above example of different metrics giving the same notion of continuity hints that our journey is not yet over: metric spaces is not the ultimate sort of objects for which continuous maps are naturally defined. To see what is, we shall make one more step. We shall get rid of 'ε-neighborhoods' and replace them by more general 'neighborhoods'.

Definition 1.3. A set $U \subset X$ in a metric space X is called *open* if every point $a \in U$ can be surrounded by an ε-ball in U, i.e., there exists ε such that $B_\varepsilon(a) \subset U$.

(An importance of open sets follows, among other things, from the fact that to develop differential calculus in $\mathbb{R}^n$ one has to be able to consider increments of an argument of a function, thus moving in all possible direction from a given point, therefore functions under consideration have to be defined on open sets in $\mathbb{R}^n$. For us, of course, the importance of open sets consists in the fact that, as we shall see it shortly, continuity can be defined entirely in terms of open sets.)

Definition 1.4. A *neighborhood* (or *open neighborhood*) of a point a is an arbitrary open set U containing a, i.e., $a \in U$.

Theorem 1.1. *For a map of metric spaces $f\colon X \to Y$, the continuity of f at a point $x_0 \in X$ is equivalent to the following condition: for every neighborhood V of $y_0 = f(x_0)$ there is a neighborhood U of x_0 such that $f(U) \subset V$.*

Proof. " $\Rightarrow$ " Suppose f is continuous at x_0. Consider an arbitrary neighborhood V of y_0, i.e., an open set V such that $y_0 \in V$. By the definition of open sets, there is an ε-neighborhood $B_\varepsilon(y_0)$ such that $B_\varepsilon(y_0) \subset V$. By the condition of continuity, there is a δ-neighborhood $B_\delta(x_0)$ such that $f(B_\delta(x_0)) \subset B_\varepsilon(y_0)$. In particular, $f(B_\delta(x_0)) \subset V$ and we can set $U = B_\delta(x_0)$. U is a desired neighborhood of x_0. " $\Leftarrow$ " Conversely, suppose for every neighborhood V of $y_0 = f(x_0)$ there is a neighborhood U of x_0 such that $f(U) \subset V$. To check the continuity, consider an ε-neighborhood $B_\varepsilon(y_0)$. By the assumption, there is a neighborhood U of x_0 such that $f(U) \subset B_\varepsilon(y_0)$. By the definition of open sets, inside U there is a δ-neighborhood $B_\delta(x_0) \subset U$. Therefore, $f(B_\delta(x_0)) \subset f(U) \subset B_\varepsilon(y_0)$. Hence, f is continuous at x_0. $\square$

Conclusion: to speak about continuity we do not need distances. All what we need, is the notion of open sets. They can be introduced by axioms.

Let X be an abstract set. We will consider collections of subsets of X denoting them by script letters.

Definition 1.5. A collection $\mathcal{O}$ of subsets of a set X is called a *topology* on X if it satisfies the following axioms:

(T1) $\varnothing \in \mathcal{O}$ and $X \in \mathcal{O}$;

(T2) if $U_i \in \mathcal{O}$ for all $i = 1, \ldots, N$, then $U_1 \cap \ldots \cap U_N \in \mathcal{O}$;

(T3) if $U_\alpha \in \mathcal{O}$ for all $\alpha \in \mathrm{A}$ (arbitrary set of indices), then $\bigcup_\alpha U_\alpha \in \mathcal{O}$.

The elements of $\mathcal{O}$ are called *open sets* (w.r.t. the topology $\mathcal{O}$).

Example 1.3. Opens sets in any metric space X satisfy the properties T1, T2, T3. Thus every metric gives rise to a topology.

Example 1.4. One can check that all three metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ define the same topology, i.e., the same collection of open sets. This topology will be called the "natural topology" of $\mathbb{R}^n$ or the *Euclidean topology* (having in mind that the metric d_2 comes from the Euclidean structure, i.e., the scalar product of vectors in $\mathbb{R}^n$).

Remark 1.1. In infinite dimension, for the space of infinite sequences $\mathbb{R}^\infty$ (here for each given sequence $a = (a_1, a_2, \ldots)$ only finitely many terms are allowed to be nonzero), the metrics d_1, d_2 and d_∞ define *different* topologies.

The analogs of d_1, d_2 and d_∞ for functions also define different topologies (for example, on the space of continuous functions on a segment; summation in the definition of d_1 and d_2 is replaced by integration).

Definition 1.6. A *topological space* is a set together with some topology on it: $(X, \mathcal{O})$.

Notation: If necessary we use letters other than $\mathcal{O}$ for topologies, e.g., $\mathcal{F}$ or $\mathcal{E}$, or use $\mathcal{O}$ with subscripts, like $\mathcal{O}_X, \mathcal{O}_Y$. (Another commonly used notation for a topology is Greek letters such as τ or σ.) Often we denote a topological space by a single letter, like $X = (X, \mathcal{O}_X)$.

Example 1.5. As follows from the above, $\mathbb{R}^n$ has a natural structure of a topological space.

Example 1.6. Any metric space can be considered as a topological space.

Example 1.7. Consider the unit circle with center at the origin in $\mathbb{R}^2$. Denote it by S^1. It can be made a topological space by introducing a metric structure. One way of defining a distance between points $a, b \in S^1$ is to set $d(a, b)$ to the Euclidean distance d_2 between a and b in the ambient space $\mathbb{R}^2$. Alternatively, by a distance between a and b one can take the length of the shortest circular arc joining a and b. Exercise: check that both definitions of distance give the same open sets. Taking them as a topology for S^1, we make it a topological space.

Example 1.8. The previous example generalizes to the sphere S^n, which as a set is defined as the unit sphere with center at the origin in $\mathbb{R}^{n+1}$.

Consider some 'abstract' examples.

Example 1.9. A one-point set $X = \{a\}$ has only one topology. Considered with it, it is called a *singleton* or simply a *point*.

Example 1.10. Any set X can be made a topological space in two ways (which are the opposite extremes). One can consider all sets in X as open. Clearly, T1, T2 and T3 will be satisfied. This topology is called the *discrete topology* and the set X with it, a *discrete topological space*. Or one can a topology on X containing just two elements: $\varnothing$ and X. This topology is called the *indiscrete topology*.

Topologies of the previous example may look artificial, but they are important for theoretical purposes.

We are almost there! Now we can define continuous maps between topological spaces following the reformulation of continuity for maps of metric spaces given by Theorem 1.1.

Definition 1.7. A map of topological spaces $f\colon X \to Y$ is *continuous at a point* $x_0 \in X$ if for every neighborhood V of $y_0 = f(x_0) \in Y$ there is a neighborhood U of x_0 such that $f(U) \subset V$.

Unlike analysis where it may be interesting to consider points where functions cease to be continuous (for example, have a jump), in topology we are interested only in maps continuous everywhere. Such maps can be described by a remarkably simple condition.

Theorem 1.2. *A map of topological spaces $f\colon X \to Y$ is continuous at all points $x \in X$ if and only if for every open set $V \subset Y$ its preimage $f^{-1}(V)$ is open in X.*

Maps continuous at all points will be simply called *continuous*. The condition given in Theorem 1.2 can be used as an alternative definition:

Definition 1.8 (Alternative). Given topological spaces $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$, a map $f\colon X \to Y$ is *continuous* if for every $V \in \mathcal{O}_Y$ the set $f^{-1}(V)$ belongs to $\mathcal{O}_X$.

Theorem 1.3 (Properties of continuous maps). *Composition of continuous maps is continuous. For each topological space the identity map is continuous.*

Proof. Suppose there are continuous maps of topological spaces $f\colon X \to Y$ and $g\colon Y \to Z$. We shall check that the composite map $g \circ f\colon X \to Z$ is continuous. To this end, consider a set $W \in \mathcal{O}_Z$. We have to check that $(g \circ f)^{-1}(W) \in \mathcal{O}_X$. Indeed, $(g \circ f)^{-1}(W) = f^{-1}\bigl(g^{-1}(W)\bigr)$. Since g is continuous and W open, the set $g^{-1}(W)$ is open. Since f is continuous, it follows that $f^{-1}\bigl(g^{-1}(W)\bigr)$ is open. Hence $g \circ f$ is continuous. Consider now the identity map $1_X\colon X \to X$ for a given topological space X. It sends each $x \in X$ to itself. In particular, for every $U \in \mathcal{O}_X$ we have $1_X^{-1}(U) = U \in \mathcal{O}_X$. Hence 1_X is continuous. $\square$

This theorem effectively says that topological spaces and continuous maps form an algebraic structure called a category. Categories provide a unifying language for many areas of mathematics and are especially valuable for topology, so we shall briefly recall their definition.

Definition 1.9. A *category* $\mathcal{C}$ consists of a collection of *objects* (of arbitrary nature; objects can be visualized as points), denoted $\operatorname{Ob}\mathcal{C}$, and a collection of *arrows* or *morphisms* (which should be visualized as actual arrows joining points representing objects), denoted $\operatorname{Mor}\mathcal{C}$. More precisely, for each pair of objects X and Y there is a set of morphisms $\operatorname{Mor}(X, Y)$ (should be visualized as arrows from X to Y), and $\operatorname{Mor}\mathcal{C}$ is the union of disjoint sets $\operatorname{Mor}(X, Y)$ over all X, Y. The algebraic structure is given by the *composition law*: a map

$$\operatorname{Mor}(X, Y) \times \operatorname{Mor}(Y, Z) \to \operatorname{Mor}(X, Z),$$

$(f, g) \mapsto g \circ f$, for each triple X, Y, Z. It should satisfy two properties:

(C1) Associativity: for any three morphisms $f \in \operatorname{Mor}(X, Y)$, $g \in \operatorname{Mor}(Y, Z)$, $h \in \operatorname{Mor}(Z, W)$

$$h \circ (g \circ f) = (h \circ g) \circ f;$$

(C2) Identity elements: for each object $X \in \operatorname{Ob}\mathcal{C}$ there is a morphism $1_X \in \operatorname{Mor}(X, X)$ such that

$$f \circ 1_X = 1_Y \circ f = f$$

for all $f \in \operatorname{Mor}(X, Y)$.

Example 1.11. A simple example of a category is an arbitrary group G: as the set of objects one can take the set consisting of a single point x_0, so $\operatorname{Ob}\mathcal{C} = \{x_0\}$, and as the set of morphisms, the set of all group elements, so $\operatorname{Mor}\mathcal{C} = \operatorname{Mor}(x_0, x_0) = G$. Since there is only one object, there is one identity map coinciding with the identity in G: $1_{x_0} = e \in G$. Clearly, the associativity holds, by the definition of a group. (Having inverses is not used, so in fact, any monoid, i.e., semigroup with identity, will give an example of a category in this way.)

Example 1.12. The notation used in the definition of a category suggests thinking of objects as of sets, and of morphisms, as of maps of sets. This is in no way necessary: from the 'categorical viewpoint' objects are whole entities, not consisting of elements; the stress being on the properties that can be expressed in terms of composition of arrows only. However, many examples of categories arise exactly in this way, with objects being sets with some extra structure and morphisms being maps respecting this structure. The simplest example is the *category of sets* denoted $\mathcal{S}ets$. $\operatorname{Ob}\mathcal{S}ets$ is the "set" of all sets. (Warning: if you studied set theory, you know that the notion of the "set of all sets" contains a contradiction; this can be remedied by only considering sets contained in some huge "universal" set, for which all sets in question can be considered as subsets. Accordingly, the definition of the category $\mathcal{S}ets$ is modified.) For two sets the set of morphisms $\operatorname{Mor}(X, Y)$ is defined as the set of all maps from X to Y. The composition and the identities have their natural meaning.

Example 1.13. Topological spaces and continuous maps make a "subcategory" of the category $\mathcal{S}ets$. If we denote the *category of topological spaces* by $\mathcal{T}op$, then $\operatorname{Ob}\mathcal{T}op$ consists of "all" topological spaces (see the remark above about "all" sets) and for two topological spaces X, Y the set of morphisms $\operatorname{Mor}(X, Y)$ in $\mathcal{T}op$ consists of all continuous maps from X to Y. Theorem 1.3 guarantees the existence of composition and identities. (The associativity of composition is a general property of the composition of arbitrary maps.)

Example 1.14. In a similar way one can define other subcategories of the category of sets, such as the *category of groups* $\mathcal{G}roups$ (objects are "all" groups, morphisms are group homomorphisms), the *category of vector spaces* $\mathcal{V}ect$ (objects are finite-dimensional vector spaces over a fixed field, such as $\mathbb{R}$, morphisms are linear transformations), etc. There are plenty of examples of such 'concrete' categories (a category is called *concrete* if it is a subcategory of $\mathcal{S}ets$). The relation between them and 'abstract' categories is the same as the relation between groups of transformations and 'abstract' groups.

The usefulness of the category language is not seen immediately. It proves itself in the course of using it. We shall see patterns of 'categorical thinking' later on.

Definition 1.10. In an arbitrary category $\mathcal{C}$, an arrow $f \in \operatorname{Mor}(X, Y)$ is called an *isomorphism* if it is invertible, i.e., there is an arrow in the opposite direction $g \in \operatorname{Mor}(Y, X)$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$. Two objects X and Y are called *isomorphic* if there is at least one isomorphism $f \in \operatorname{Mor}(X, Y)$.

One can see particular cases of this notion in the concepts of a group isomorphism, an isomorphism of vector spaces, etc. For the category of topological spaces we arrive to the following particular case of Definition 1.10:

Definition 1.11. A map of topological spaces $f\colon X \to Y$ is a *homeomorphism* if three conditions are satisfied: f is continuous, f is invertible, i.e., there is a map $g\colon Y \to X$ such that $f \circ g = 1_Y$ and $g \circ f = 1_X$, and the inverse map $g = f^{-1}$ is also continuous. Topological spaces X and Y are called *homeomorphic* if there is at least one homeomorphism $f\colon X \to Y$. Notation: $X \cong Y$.

Now we can see that the goal set in the beginning is fulfilled: the definition of homeomorphism of topological spaces is exactly what we were looking for. We shall use *topological equivalence* as the synonym for homeomorphism.

Theorem 1.4. *Topological equivalence or homeomorphism is indeed an equivalence relation for topological spaces.*

Proof. We have to check that reflexivity, symmetry and transitivity hold for the relation $\cong$. Indeed, to show that $X \cong X$, for all X, we use the identity map: $1_X\colon X \to X$ is continuous and invertible, the inverse being 1_X itself, so it is also continuous. To show that $X \cong Y$ implies $Y \cong X$, consider a homeomorphism $f\colon X \to Y$. Its inverse $f^{-1}\colon X \to Y$ is also a homeomorphism, as it is continuous, invertible, the inverse being f therefore continuous. Finally, if $X \cong Y$ and $Y \cong Z$, there are homeomorphisms $f\colon X \to Y$ and $g\colon Y \to Z$. Their composition is a homeomorphism, as immediately follows from Theorem 1.3. Hence $X \cong Z$. □

Remark 1.2. One can see that there was nothing peculiar for topological spaces and continuous maps in the proof of Theorem 1.4. We only used the fact that homeomorphisms are isomorphisms in the category $\mathcal{T}op$. In any category the notion of an isomorphism gives rise to an equivalence relation on the set of objects. (Examples: isomorphisms of groups, vector spaces, rings, etc.)

We shall give some simple example of homeomorphism of topological spaces. More examples will appear later.

Example 1.15. Consider the real line $\mathbb{R}$ and an arbitrary segment (α, β). Both taken with the natural topology coming from the metric $d(a, b) = |a-b|$. Claim: $\mathbb{R} \cong (\alpha, \beta)$. Indeed, without loss of generality we may take $(\alpha, \beta) = (-1, 1)$, since any two finite intervals are clearly homeomorphic (check!). As a desired homeomorphism one can take the map $f\colon\ (\alpha, \beta) \to \mathbb{R}$, $f(x) = \tan \frac{\pi x}{2}$. For arbitrary (α, β) the statement follows by transitivity.

Example 1.16. Check that for any n the space $\mathbb{R}^n$ is homeomorphic to the open ball $B = B_1(O) = \{\boldsymbol{x} \in \mathbb{R}^n \mid ||\boldsymbol{x}|| < 1\}$. Here $||\boldsymbol{x}||^2 = (x_1)^2 + \dots (x_n)^2$ is the Euclidean norm.

Example 1.17. By using the stereographic projection with center at the 'north pole' $N = (0, \dots, 0, 1)$ onto the plane $x_{n+1} = 0$, one can check that the sphere without a point, $S^n \setminus \{N\}$ is homeomorphic to $\mathbb{R}^n$.

Example 1.18. Later we shall see that S^n and $\mathbb{R}^n$ are **not** homeomorphic.

Example 1.19. The 2-dimensional sphere S^2 and the surface of a bagel (the 2-torus) T^2 are **not** homeomorphic. This can be shown, for example, by considering closed curves without self-intersections on both surfaces. It is intuitively clear that any such curve on S^2 bounds a piece of surface, while on T^2 there are closed curves (for example, a circle obtained by cutting the torus by a plane through its axis of symmetry) not bounding anything. To make this argument precise one has, however, to develop certain topological technique.

We shall conclude the section by a discussion of one technical notion that will be used in the future. Let X be an abstract set. A *base* on a set X is a collection of subsets $\mathcal{B}$ with the properties:
(B1) $\bigcup_{B \in \mathcal{B}} B = X$;
(B2) if $B_1 \in \mathcal{B}$ and $B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ is a union of elements of $\mathcal{B}$.

Example 1.20. Every topology $\mathcal{O}$ on X satisfies the axioms of a base.

However, not every base $\mathcal{B}$ is a topology (for example, it is not required that finite intersections belong to the same family $\mathcal{B}$; they have to be just the unions of elements of $\mathcal{B}$). We shall use bases for constructing topologies.

Proposition 1.1. *Given a set X, if $\mathcal{B}$ is a base, then the collection of all unions of elements of $\mathcal{B}$ is a topology.*

Proof. Denote the collection of all unions of elements of $\mathcal{B}$ by $\mathcal{O}$. We have to show that $\mathcal{O}$ is a topology. Check T1, T2 and T3. Consider T1. By B1, $X = \bigcup_{B\in\mathcal{B}} B$, hence $X \in \mathcal{O}$. The empty set $\varnothing$ is the union of the empty collection of elements of $\mathcal{B}$. Therefore T1 is satisfied. Consider T2. Suppose $U = \bigcup B_i$ and $V = \bigcup_\mu B_\mu$ where $B_i, B_\mu \in \mathcal{B}$. Then

$$U \cap V = \left(\bigcup_i B_i\right) \cap \left(\bigcup_\mu B_\mu\right) = \bigcup_{i,\mu} B_i \cap B_\mu = \bigcup_{i,\mu,\alpha} B_{i\mu\alpha}$$

where $B_{i\mu\alpha} \in \mathcal{B}$. We used here the axiom B2 for the base $\mathcal{B}$. Hence $U\cap V \in \mathcal{O}$, and T2 is satisfied. Finally, consider T3. For an arbitrary family $U_\alpha = \bigcup_\mu B_{\alpha\mu} \in \mathcal{O}$ we have

$$\bigcup_\alpha U_\alpha = \bigcup_{\alpha\mu} B_{\alpha\mu}\,,$$

which is an element of $\mathcal{O}$. Therefore T3 is also satisfied, and $\mathcal{O}$ is a topology. □

We shall sometimes use the notation $\overline{\mathcal{B}}$ for the topology consisting of all unions of elements of a base $\mathcal{B}$. It is called the topology *generated* by the base $\mathcal{B}$. On the other hand, if a topology $\mathcal{O}$ is already given, and $\mathcal{O} = \overline{\mathcal{B}}$, we say that $\mathcal{B}$ is a base for the topology $\mathcal{O}$.

Example 1.21. Let X be a metric space. The collection of all open balls is a base for the topology on X considered as a topological space. (There are two statements: open balls make a base, and this base generate the topology. The crucial fact is that every open set U is the union of certain open balls. This can be seen as follows: take for every point $x \in U$ an open ball $B_x \subset U$ containing x; then $U = \bigcup_x B_x$.)

We shall see more examples of bases in the future. The point of introducing bases is to be able to describe topological properties using a "smaller" number of open sets compared with the collection of all open sets (i.e., the topology).

Proposition 1.2. *Given topological spaces $X = (X, \mathcal{O}_X)$ and $Y = (Y, \mathcal{O}_Y)$. Suppose $\mathcal{B}_Y$ is a base for the topology $\mathcal{O}_Y$. A map $f\colon X \to Y$ is continuous if and only if for every $B \in \mathcal{B}_Y$ its preimage $f^{-1}(B)$ belongs to $\mathcal{O}_X$.*

Proof. Since $\mathcal{B}_Y \subset \mathcal{O}_Y$, if f is continuous, then $f^{-1}(B) \in \mathcal{O}_X$ for every $B \in \mathcal{B}_Y$. Conversely, suppose this condition holds. Check continuity. For an arbitrary open set $V \in \mathcal{O}_Y$ we have $V = \bigcup B_i$ where $B_i \in \mathcal{B}_Y$, hence $f^{-1}(V) = f^{-1}\Big(\bigcup B_i\Big) = \bigcup f^{-1}(B_i)$, which is open, since each $f^{-1}(B_i)$ is open by the assumption. Hence f is continuous. □