§3 Fundamental topological properties

3.1 Closed sets

Consider a topological space $X = (X, \mathcal{O}_X)$.

Definition 3.1. A set $C \subset X$ is *closed* if $C^c = X \setminus C$ is open.

Example 3.1. A point, a plane in $\mathbb{R}^n$ (directly from definition: it is easy to see that every point in the complement can be surrounded by a small open ball, hence the complement is open); S^n in $\mathbb{R}^{n+1}$ (to be proved later).

Example 3.2. For discrete topology: all sets are closed.

Example 3.3. For indiscrete topology: the closed sets are $X, \varnothing$.

Example 3.4. For cofinite topology: the closed sets are the finite sets and X.

Theorem 3.1. *Properties of closed sets (dual to those of opens sets):*
(T1°) $\varnothing$ *and* X *are closed;*
(T2°) *if* C_i *are closed for all* $i = 1, \ldots, N$, *then* $C_1 \cup \ldots \cup C_N$ *is closed ("finite unions");*
(T3°) *if* C_α *are closed for all* $\alpha \in \mathrm{A}$ *(arbitrary set of indices), then* $\bigcap_\alpha C_\alpha$ *is closed ("arbitrary intersections").*

Proof. Follows from the properties of open sets and De Morgan's formulae. □

Theorem 3.2. *A map is continuous if and only if the preimage of every closed set is closed.*

Proof. Consider topological spaces X and Y, and a map $f\colon X \to Y$. "$\Rightarrow$" Suppose f is continuous. Take a closed set $C \subset Y$ and consider $f^{-1}(C) \subset X$. We have $C = Y \setminus V$ for some $V \in \mathcal{O}_Y$. Hence $f^{-1}(C) = f^{-1}(Y \setminus V) = \{x \in X \mid f(x) \notin C\} = X \setminus f^{-1}(V)$. By the continuity of f, the set $f^{-1}(V)$ is open, hence the set $X \setminus f^{-1}(V)$ is closed. "$\Leftarrow$" Conversely, suppose that $f^{-1}(C) \subset X$ is closed for every closed $C \subset Y$. Take an arbitrary $V \in \mathcal{O}_Y$. We have $V = Y \setminus C$ for some closed C, and $f^{-1}(V) = f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$, similarly to the above. By the assumption $f^{-1}(C)$ is closed, hence its complement is open, and it is $f^{-1}(V)$. Hence f is continuous. □

Example 3.5. Consider in $\mathbb{R}^n$ a set C specified by equations:

$$f_1(x_1, \ldots, x_n) = 0,$$
$$\vdots$$
$$f_p(x_1, \ldots, x_n) = 0.$$

Suppose the left-hand sides are continuous functions. Then C is closed. Indeed, we can consider $F\colon \mathbb{R}^n \to \mathbb{R}^p$ by setting

$$F(x_1, \ldots, x_n) = \big(f_1(x_1, \ldots, x_n), \ldots, f_p(x_1, \ldots, x_n)\big).$$

It is continuous by properties of product topology. Notice that $C = F^{-1}(\{0\})$. Therefore C is closed as the preimage of a closed set under a continuous map.

There are plenty of examples.

Example 3.6. S^n is a closed subset of $\mathbb{R}^{n+1}$. Matrix groups such as $O(n)$, $SL(n)$ specified by matrix equations in the space of all matrices are closed subsets.

There is an interesting operation producing a closed set from an arbitrary given set $A \subset X$. Consider all closed sets containing A. Define the *closure* of A, notation $\overline{A}$, as their intersection:

$$\overline{A} := \bigcap_{\substack{C \text{ is closed} \\ A \subset C}} C\,.$$

Clearly, $A \subset \overline{A}$ (because A is contained in all sets of which we take the intersection), $\overline{A}$ is closed (as the intersection of a family of closed sets) and any closed set containing A contains $\overline{A}$. Hence $\overline{A}$ is the smallest closed set containing A and can be defined as such.

Example 3.7. Consider the set $[0, 1) \subset \mathbb{R}$. Clearly, $[0, 1)$ is not closed and its closure is $[0, 1]$ as the smallest closed set containing $[0, 1)$.

Remark 3.1. If C is closed, obviously $\overline{C} = C$. This and other properties of the "closure operator" $A \mapsto \overline{A}$ made axioms can be used for an alternative axiomatic approach to topological spaces. Such an approach was suggested by the Polish topologist Kuratowski in 1920s.

Remark 3.2. There is a notion 'dual' to the notion of closure. Namely, one can consider the largest open subset of a given set. It is called the *interior* of a set, notation: $\mathring{A}$. Notice that for the closure we take supersets of A and for the interior, subsets (check that doing it other way round would give uninteresting answers). In particular examples the closure of a set can be the whole space (such sets are called *dense*). Similarly, in particular examples the interior of a set can be empty.

In analysis they give an alternative description of a closed set in a metric space: a set is closed if it contains all limits of convergent sequences (that is, if a sequence of elements of $A \subset X$ converges, then its limit $a^* \in X$ has to be in A). Accordingly, the operation of taking closure in this language becomes transparent: in addition to the points of A, the set $\overline{A}$ contains all lacking limits of sequences.

3.2 Hausdorff property

Definition 3.2. A topological space $X = (X, \mathcal{O}_X)$ is *Hausdorff* (or possesses the *Hausdorff property*) if for every two points $a, b \in X$ such that $a \neq b$ there are open neighborhoods U_a and U_b of a and b respectively such that $U_a \cap U_b = \varnothing$.

Shortly, we say that in a Hausdorff space distinct points can be 'separated' by disjoint neighborhoods.

Remark 3.3. There are several "separation properties" considered in general topology. The Hausdorff property is the strongest and the most important of them.

Example 3.8. The real line $\mathbb{R}$ is Hausdorff. Indeed, for real numbers $a \neq b$ we can take ε-neighborhoods $(a-\varepsilon, a+\varepsilon)$ and $(b-\varepsilon, b+\varepsilon)$ with $\varepsilon < \frac{1}{2}|a-b|$ and they will not intersect.

This immediately generalizes to arbitrary metric spaces.

Theorem 3.3. *Every metric space is Hausdorff.*

Proof. Suppose we are given $a \neq b$ in a metric space X. Set $\varepsilon = \frac{1}{3}\, d(a, b)$ and consider the open balls $B_\varepsilon(a)$ and $B_\varepsilon(b)$. Claim: they do not intersect. Indeed, suppose there is a point $c \in B_\varepsilon(a) \cap B_\varepsilon(b)$. Then, by the triangle inequality

$$d(a, b) \leqslant d(a, c) + d(c, b) < \varepsilon + \varepsilon = \frac{2}{3}\, d(a, b),$$

which is a contradiction! Hence no such c can exist. □

In particular, $\mathbb{R}^n$ is Hausdorff.

Let us give examples of non-Hausdorff spaces.

Example 3.9. An indiscrete space $X, \mathcal{O}_X = \{X, \varnothing\}$ is non-Hausdorff. Indeed, there is only one nonempty open set, therefore it is not possible to separate two points by disjoint open neighborhoods.

Example 3.10. $\mathbb{R}$ with the cofinite topology is non-Hausdorff. Indeed, one can notice that any two open sets have nonempty intersection (suppose there are two disjoint open sets, then the union of their complements, which are finite, should be $\mathbb{R}$ — a contradiction!).

Typical "well-behaved" topological spaces are Hausdorff. Exceptions from this rule are topological spaces arising in algebraic geometry. Closed sets there is given by systems of algebraic equations. For example, on a line there is only one equation, which can have only a finite number of roots, so it is the case of the cofinite topology. In general, the topology will not be cofinite, but will still be non-Hausdorff. Another source of non-Hausdorff spaces, as we shall see later, are certain 'spaces of orbits'.

Proposition 3.1. *Every point in a Hausdorff space is closed (more precisely, all one-point sets are closed).*

Proof. Indeed, by the Hausdorff property, in the complement $X \setminus \{a\}$ of an arbitrary point $a \in X$ every point $b \in X \setminus \{a\}$ has a neighborhood not containing a (this follows from the Hausdorff property and is itself a weaker "separation condition"). Hence $X \setminus \{a\}$ is open and $\{a\}$ is closed. □

(We have implicitly used this when speaking about closed sets in $\mathbb{R}^n$ in the previous subsection.)

Theorem 3.4. *A subspace of a Hausdorff space is Hausdorff.*

Proof. Suppose $A \subset X$ is a subspace of a Hausdorff space. Consider $a, b \in A$ such that $a \neq b$. Since X is Hausdorff, there are $U, V \in \mathcal{O}_X$ s.t. $a \in U$, $b \in V$ and $U \cap V = \varnothing$. Consider $U \cap A \in \mathcal{O}_A$, $V \cap A \in \mathcal{O}_A$. We have $a \in U \cap A$, $b \in V \cap A$, and $(U \cap A) \cap (V \cap A) = \varnothing$ (sketch a picture!). Hence A is Hausdorff. □

Corollary 3.1. *If a topological space X can be embedded into $\mathbb{R}^N$ as a subspace, X must be Hausdorff.*

Theorem 3.5. *The product of Hausdorff spaces is Hausdorff.*

Proof. Let X and Y be Hausdorff. Consider $X \times Y$. Suppose $(x, y) \neq (x', y')$ in $X \times Y$. Then either $x \neq x'$ or $y \neq y'$. Assume the former. Then by the Hausdorff property of X, there are disjoint open neighborhoods U_x and $U_{x'}$ of the respective points. Take $U_x \times Y$ and $U_{x'} \times Y$. They are disjoint open neighborhoods for (x, y) and (x', y'). If $x = x'$, but $y \neq y'$, then we can apply the same argument using the Hausdorff property of Y. □

What happens with Hausdorff spaces under identification?

Example 3.11. Consider the following equivalence relation on $\mathbb{R}^2 \setminus \{0\}$: two vectors $\boldsymbol{v}, \boldsymbol{u}$ are equivalent if $\boldsymbol{u} = \lambda \boldsymbol{v}$ where $\lambda > 0$. Clearly, $(\mathbb{R}^2 \setminus \{0\})/\sim \cong S^1$ (sketch a picture). We see that here the identification space is Hausdorff.

This is not always the case, as shown by a slight modification of the same example.

Example 3.12. Consider the same equivalence relation as above on the whole $\mathbb{R}^2$ (including the zero vector). Clearly, the equivalence classes of nonzero vectors will be as above, and the equivalence class of 0 will contain only 0. Hence, $\mathbb{R}^2/\sim$ can be identified with $S^1 \cup \{0\}$ as a set. What about topology? Suppose an open set V in the identification topology contains 0. That means that $p^{-1}(V)$ is open and contains 0, hence it contains an open disk with center at the origin. Hence $p(p^{-1}(V)) = V$ is the whole $S^1 \cup \{0\}$ (as the classes of non-zero vectors of length $< \varepsilon$ will give the whole circle S^1). It follows that 0 in our identification space cannot be separated from any point of the circle, thus this space is non-Hausdorff.

So we have to be cautious with identification spaces. They can be non-Hausdorff even if the initial space is Hausdorff.

In good cases we can still get Hausdorff spaces after an identification. One more example:

Example 3.13. The Klein bottle K^2 is Hausdorff. Indeed, consider two distinct points of K^2. If they are represented by points inside the square, they clearly can be separated by small open disks. If a point of K^2 is represented by a point on the boundary of the square, as the corresponding neighborhood one can take the image of two "symmetric" small half-disks (consider a picture!), which in K^2 will be glued to a give a disk (more precisely, an open set homeomorphic to an open disk).

Conclusion: "good" spaces are Hausdorff; such are all subspaces of $\mathbb{R}^n$ and all metric spaces; subspaces and products of Hausdorff spaces are Hausdorff. Identification spaces of a Hausdorff space are not necessarily Hausdorff (though might be).

We should also add that, as one will clearly expect, being Hausdorff is a 'topological' property, holding (or otherwise) for all homeomorphic spaces:

Theorem 3.6. *If X is Hausdorff and $X \cong Y$, then Y is Hausdorff.*

Proof. Indeed, let $f\colon X \to Y$ and $g\colon Y \to X$ be mutually inverse homeomorphisms. Suppose $y_1 \neq y_2$. Consider $x_1 = g(y_1)$ and $x_2 = g(y_2)$. Clearly, $x_1 \neq x_2$ (our maps are invertible!). Since X is Hausdorff, there are open neighborhoods $U_1 \ni x_1$ and $U_2 \ni x_2$ such that $U_1 \cap U_2 = \varnothing$. Take open neighborhoods $V_1 = f(U_1) = g^{-1}(U_1)$ and $V_2 = f(U_2) = g^{-1}(U_2)$ of y_1 and y_2, respectively. They are as desired (check that the intersection $V_1 \cap V_2$ is empty!). □

3.3 Compactness

3.3.1 General theory

Recall that a *cover* of a topological space X is a family of subsets $\mathfrak{A} = (A_\alpha)$ such that $X = \bigcup A_\alpha$. The cover $\mathfrak{A}$ is *open* if all sets A_α are open.

Definition 3.3. A topological space X is *compact* if every open cover of X contains a finite subcover.

Example 3.14. A discrete space is compact if and only if it is finite. Indeed, consider an open cover consisting of single-point sets: $X = \bigcup_{x\in X}\{x\}$. (They are open because X is discrete.) The possibility to extract a finite subcover, $X = \{x_1\} \cup \ldots \cup \{x_N\}$, is clearly the condition $\operatorname{card} X < \infty$.

Example 3.15. Every indiscrete space is compact (indeed, what is an open cover in this case?).

Theorem 3.7. *A continuous image of a compact space is compact.*

Proof. Consider a continuous map $f\colon X \to Y$. Let $Y = \bigcup V_\alpha$ be an open cover of Y. Take the preimages: $f^{-1}(V_\alpha)$. They are open, because f is continuous. We obviously have $X = \bigcup f^{-1}(V_\alpha)$, because each point of X is mapped to some point of Y, which belongs to one of the sets V_α. That means that each point of X is in some $f^{-1}(V_\alpha)$. So we have an open cover. Since X is compact, we can extract a finite subcover: $X = f^{-1}(V_{\alpha_1}) \cup \ldots \cup f^{-1}(V_{\alpha_N})$. Applying f to this equality and recalling that the image of the union is the union of the images and $f(f^{-1}(V_\alpha)) \subset V_\alpha$, we obtain $Y = V_{\alpha_1} \cup \ldots \cup V_{\alpha_N}$. □

Corollary 3.2. *If X is compact and $X \cong Y$, then Y is compact.*

Proof. Indeed, $Y = f(X)$ where $f\colon X \to Y$ is a homeomorphism. □

In other words, compactness is a 'topological property', simultaneously holding, or not, for all topologically equivalent spaces.

Corollary 3.3. *All identification spaces of a compact space are compact.*

Proof. Indeed, an identification space is the image of the canonical projection, which is continuous. □

What is the relation of compactness with subspaces? A subspace of a compact space is not necessarily compact. We shall see examples of this later. However, the situation is different for **closed** subspaces.

Theorem 3.8. *A closed subspace of a compact space is compact.*

Remark 3.4. The notion of compactness for subspaces can be reformulated in the following convenient form. Suppose $A \subset X$ is a subspace of a topological space X; an *open cover of A in X* is a collection $\mathfrak{U} = (U_\alpha)$ where all sets $U_a \subset X$ are open in X and $A \subset \bigcup U_\alpha$. Clearly, open covers of A in X are in one-one correspondence with open covers of A as a topological space (with the subspace topology): one can take $A \cap U_\alpha$. Hence compactness for subspaces of X can be formulated in terms of open covers in X: *a subspace $A \subset X$ is compact if and only if every open cover of A in X, $A \subset \bigcup U_\alpha$, contains a finite subcover: $A \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_N}$.*

Proof of Theorem 3.8. Let $C \subset X$ be a closed subspace. Consider an open cover of C in X: $A \subset \bigcup U_\alpha$. Since C is closed, $V = X \setminus C$ is open, and V together with all U_α will give an open cover of X. By the compactness of X, we can extract a finite subcover: $X = U_{\alpha_1} \cup \ldots \cup U_{\alpha_N} \cup V$ or $X = U_{\alpha_1} \cup \ldots \cup U_{\alpha_N}$. Throwing away V if present, we obtain a finite cover of C in X, which is a subcover of (U_α). Hence C is compact. □

There is a useful technical lemma.

Lemma 3.1. *Checking compactness of a topological space X, it is possible to consider only the covers consisting of elements of a fixed base of the topology of X.*

Proof. Left as an exercise. □

Theorem 3.9. *A product of compact spaces is compact.*

Proof. [1] We shall prove this for the product of two compact space. Suppose X, Y are compact. To prove that the product space $X \times Y$ is compact, consider an arbitrary cover of $X \times Y$ by open sets of the form $U_\alpha \times V_\alpha$, where $U_\alpha \in \mathcal{O}_X$, $V_\alpha \in \mathcal{O}_Y$. By Lemma 3.1 it is sufficient to consider only such open covers. Let $x \in X$ be an arbitrary point of X. Consider the subspace $\{x\} \times Y \subset X \times Y$. It is compact (as homeomorphic to Y), hence

$$\{x\} \times Y \subset U_{\alpha_1(x)} \times V_{\alpha_1(x)} \cup \ldots \cup U_{\alpha_N(x)} \times V_{\alpha_N(x)}.$$

Consider the intersection $U^x := U_{\alpha_1(x)} \cap \ldots \cap U_{\alpha_N(x)}$. Notice that it is an open subset of X. We have, clearly,

$$U^x \times Y \subset U_{\alpha_1(x)} \times V_{\alpha_1(x)} \cup \ldots \cup U_{\alpha_N(x)} \times V_{\alpha_N(x)}.$$

On the other hand, the collection of all U^x (for all $x \in X$) is an open cover of X. Since X is compact, there is a finite subcover: $X = U^{x_1} \cup \ldots \cup U^{x_{N'}}$. It follows that

$$X \times Y = U^{x_1} \times Y \cup \ldots \cup U^{x_{N'}} \times Y \subset \Big(U_{\alpha_1(x_1)} \times V_{\alpha_1(x_1)} \cup \ldots \cup U_{\alpha_N(x_1)} \times V_{\alpha_N(x_1)}\Big) \cup \ldots \cup \Big(U_{\alpha_1(x_{N'})} \times V_{\alpha_1(x_{N'})} \cup \ldots \cup U_{\alpha_N(x_{N'})} \times V_{\alpha_N(x_{N'})}\Big).$$

□

To conclude our general discussion of compact spaces, let us make a statement which is almost obvious:

Theorem 3.10. *If topological spaces X and Y are homeomorphic, and X is compact, then Y is also compact.*

Proof. Indeed, suppose $f\colon X \to Y$ is a homeomorphism. Then $Y = f(X)$, and the statement follows from Theorem 3.7. □

In other words, compactness is a 'topological' property, which holds, or otherwise, simultaneously for all topologically equivalent spaces.

[1] Required only for MSc students.

3.3.2 Compactness for metric spaces and subspaces of $\mathbb{R}^n$

Theorem 3.11. *If a metric space is compact, then it is bounded.*

Proof. Indeed, choose a point a and consider the cover $\mathfrak{B} = (B_n(a))$ by open balls $B_n(a)$ with center at a and radii $n = 1, 2, \ldots$. We have

$$B_1(a) \subset B_2(a) \subset \ldots .$$

A possibility to extract a finite subcover will mean that the whole space is a ball $B_N(a)$, for some N. □

The following properties concerning compactness and metric spaces are an extra material not included in the course. They give a general criterion of compactness for metric spaces.

Theorem 3.12. *If a metric space is compact, then it is complete.* (Notice that completeness is not a topological property!)

Proof. Should be given in the analysis course. □

Theorem 3.13 (Hausdorff criterion)**.** *A metric space is compact if and only if it is complete and for any $\varepsilon > 0$ can be covered by a finite number of ε-balls.* (This is stronger than being bounded.)

Proof. Omitted. □

Now let us turn from general metric spaces to subspaces of $\mathbb{R}^n$.

Lemma 3.2 (Heine–Borel Lemma). *$I^n \subset \mathbb{R}^n$ is compact.*

Proof. By subdivision. Suppose this is not true, i.e., there exists an open cover of the cube such that there is no finite subcover. Fix this cover, denote it $\mathfrak{U}$. We shall get a contradiction. The method is subdividing our cube into increasingly smaller cubes. Divide the side by 2 and consider the resulting 2^n cubes with side 1/2. $\mathfrak{U}$ is an open cover for each of them. If for every small cube there is a finite subcover, then together they will give a finite subcover for the whole cube, which we assume does not exist. Hence, there is no finite subcover for at least one small cube. Denote it C_1. (We denote the original cube C_0.) Divide the side of C_1 by 2. We get even smaller cubes (with side 1/4). For the same reason, at least one of them cannot be covered by a finite number of elements of $\mathfrak{U}$. Denote it C_2. (We choose one, if there are several.) Continuing in this way we will obtain a sequence of nested cubes

$$C_0 \supset C_1 \supset C_2 \supset \ldots$$

with sides $1, 1/2, 1/4, \ldots$ having the property that none of them can be covered by a finite number of elements of the cover $\mathfrak{U}$. On the other hand, from one of the properties of real numbers (the intersection of a nested sequence of closed segments is not empty) follows that $\bigcap C_k \neq \varnothing$. Take a point $a \in \bigcap C_k$. Clearly, $a \in U$ for some $U \in \mathfrak{U}$, because $\mathfrak{U}$ is a cover for $[0,1]^n$. Since U is open, it contains an ε-ball with centre a. Obviously, for N large enough, $C_N \subset B_\varepsilon(a)$, because $a \in C_N$ and the side of the cube can be made as small as we wish. Hence, C_N is covered by a single element U from $\mathfrak{U}$, which is a contradiction to our choice of the cubes $C_0, C_1, C_2, \ldots$. It follows that our assumption was wrong, and $C_0 = [0,1]^n$ *can* be covered by a finite number of elements of an arbitrary open cover $\mathfrak{U}$, i.e., that $[0,1]^n$ is compact. □

Corollary 3.4. *Any closed cube in* $\mathbb{R}^n$ *is compact.*

Theorem 3.14 (Heine–Borel Theorem)**.** *Every closed and bounded subspace in* $\mathbb{R}^n$ *is compact. Conversely, every compact subspace in* $\mathbb{R}^n$ *is closed and bounded.*

Proof. Suppose a subspace $A \subset \mathbb{R}^n$ is a closed and bounded subset. Since A is bounded, there is a cube $C = [-R, R]^n$ such that $A \subset C$. The subset $A \subset C$ is closed. As follows from the Heine–Borel Lemma, the cube C is compact. Hence, A is compact as a closed subspace of a compact space. Conversely, assume that a subspace $A \subset \mathbb{R}^n$ is compact. A must be bounded, as a compact metric space. It remains to prove that A is a closed subset of $\mathbb{R}^n$. This will be done later (see below after Theorem 3.15). □

There are plenty of examples of compact and non-compact spaces, which can obtained using the Heine–Borel Theorem.

Example 3.16. $\mathbb{R}^n$ itself is non-compact (unbounded).

Example 3.17. S^n, which we consider as the unit sphere with center at the origin in $\mathbb{R}^{n+1}$, is compact. Indeed, it is bounded (easy to show: all distances are bounded by 2) and it is closed in $\mathbb{R}^{n+1}$ as a set specified by an equation with continuous left-hand side (see Example 3.6).

Example 3.18. Among matrix groups compact are, for example, $O(n)$, $SO(n)$, $U(n)$ and $SU(n)$. All of them are closed (as sets specified by continuous equations in the spaces of all matrices) and bounded, which can be shown directly. At the same time, non-compact are, for example, $GL(n)$, $SL(n)$, because they are unbounded.

Example 3.19. An open interval $(0,1)$ is non-compact. Though it is bounded, it is not closed as a subset of $\mathbb{R}$.

3.3.3 Compactness and Hausdorff spaces

Lemma 3.3 ("Separation of a point from a compact subspace")**.** *Suppose* X *is Hausdorff. Let* $K \subset X$ *be compact,* $a \notin K$*. Then there are open sets* $U_K \supset K$ *and* $U_a \ni a$ *such that* $U_K \cap U_a = \varnothing$*.*

Proof. Since X is Hausdorff, for all points $b \in K$ we can find open sets $U_b \ni b$ and $V_b \ni a$ such that $U_b \cap V_b = \varnothing$. We get an open cover of K by all such U_b. Since K is compact, there is an open subcover: $K \subset U_{b_1} \cup \ldots \cup U_{b_N}$ where $U_{b_i} \cap V_{b_i} = \varnothing$. Denote $U_K := U_{b_1} \cup \ldots \cup U_{b_N}$. Take $V_{b_1} \cap \ldots \cup V_{b_N} =: U_a$. U_a contains a and is open. Moreover, $U_a \cap U_{b_i} = \varnothing$ for all i, hence $U_a \cap U_K = \varnothing$, as required. □

Theorem 3.15. *Every compact subspace of a Hausdorff space is closed.*

Proof. Take $K \subset X$. Consider $X \setminus K$. A point from $X \setminus K$ can be separated from K by an open set. Hence every point of $X \setminus K$ has an open neighborhood contained in $X \setminus K$. Hence $X \setminus K$ is open, i.e., K is closed. □

As a corollary, we see that *every compact subspace of* $\mathbb{R}^n$ *is a closed subset*, which completes the above proof of the Heine–Borel theorem.

Theorem 3.16 (Homeomorphism theorem)**.** *If a continuous map from a compact space to a Hausdorff space is invertible, then it is a homeomorphism.*

Proof. Consider $f\colon X \to Y$ so that X is compact, Y Hausdorff, f continuous and invertible. Consider the map $f^{-1}\colon Y \to X$. We have to check that it is continuous. For this sake, we shall check that for any closed set $C \subset X$, its preimage under f^{-1}, that is $(f^{-1})^{-1}(C)$, is closed in Y. Notice that $(f^{-1})^{-1}(C) = f(C)$. Now, $C \subset X$ is compact as a closed subset of a compact space; therefore $f(C) \subset Y$ is compact as a continuous image of a compact space. Since Y is Hausdorff, it follows that $f(C)$ is closed, as claimed. Hence f^{-1} is continuous. □

Corollary 3.5. *Given a continuous map $f\colon X \to Y$, where X is compact, Y Hausdorff, so that f is one-to-one (injective). Then f is a homeomorphism onto $f(X)$.*

Applications of this theorem (and the corollary) are related with the case $Y = \mathbb{R}^n$. For many reasons it is desirable to embed an abstract topological X into some $\mathbb{R}^n$, with a suitable n, as a subspace.

Example 3.20. Consider the projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$. Both are defined as abstract topological spaces. However, it is not difficult to construct continuous and injective maps of each of the spaces to $\mathbb{R}^N$, for a sufficiently large N. (One can associate with a line in $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$ a linear operator on $\mathbb{R}^{n+1}$ or $\mathbb{C}^{n+1}$, respectively, in such a way that it will give a continuous injection of $\mathbb{R}P^n$ or $\mathbb{C}P^n$ to $\mathrm{Mat}(n+1,\mathbb{R}) \cong \mathbb{R}^{(n+1)^2}$ or $\mathrm{Mat}(n+1,\mathbb{C}) \cong \mathbb{R}^{2(n+1)^2}$. Since $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are compact (as continuous images of spheres), the corollary from the Homeomorphism Theorem is applicable. Hence $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are homeomorphic to subspaces of a Euclidean space.

In particular, it follows that $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are Hausdorff.

3.4 Connectedness and path-connectedness

[Under construction]

Definition 3.4. A topological space X is called *connected* if it cannot be presented as the union of two disjoint non-empty sets. Otherwise, X is *disconnected.*

In more detail, X is disconnected if there are $U, V \in \mathcal{O}_X$ such that $U \neq \varnothing$, $V \neq \varnothing$, $X = U \cup V$ and $U \cap V = \varnothing$. It is connected if no such U and V exist.

Example 3.21. If X is discrete, then X is connected only if X is a singleton. If $\operatorname{card} X > 1$, then it is disconnected. Indeed, if there are more than two points in X, we can write $X = \{x_0\} \cup (X \setminus \{x_0\})$. Otherwise there is only one non-empty set $\{x_0\} = X$.

Example 3.22. An indiscrete space X is always connected. (There are not too many open sets in X!) Indeed, the only nonempty open set is X, hence its complement is the empty set. It follows that it is not possible to present X as $X = U \cup V$ where U, V are open, nonempty, and $U \cap V = \varnothing$.

Theorem 3.17. *The continuous image of a connected space is connected.*

Proof. Consider a continuous map $f\colon X \to Y$ where X is connected. Suppose $f(X)$ is disconnected (as a subspace of Y). That means that $f(X) = A \cup B$ where A and B are nonempty open sets in $f(X)$ so that $A \cap B = \varnothing$. Consider $X = f^{-1}(f(X)) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$. We have: $f^{-1}(A)$ and $f^{-1}(B)$ are open (since f is continuous) and nonempty (because $A \subset f(X)$, $B \subset f(X)$), and $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = \varnothing$. Contradiction with the connectedness of X! □

Corollary 3.6. *If X is connected and $X \cong Y$, then Y is connected.*

Proof. Indeed, $Y = f(X)$ where $f\colon X \to Y$ is a homeomorphism. □

In other words, connectedness is a 'topological property'.

Corollary 3.7. *All identification spaces of a connected space are connected.*

Proof. Indeed, an identification space is the image of the canonical projection, which is continuous. □

Theorem 3.18. *If X, Y are connected, then $X \times Y$ is connected.*

(Proof required for MSc only)

Lemma 3.4. *Every segment $[a, b]$ is connected.*

(Proof required for MSc only)

Definition 3.5. A *path* in a topological space X is a continuous map

$$\gamma\colon [0,1] \to X.$$

Remarks: paths are also called 'parametrized curves'. A path is not a subset of X, but a map. Often instead of $[0,1]$ an arbitrary segment $[a,b]$ is considered (this has advantages and disadvantages).

Paths can be *composed.* Namely, define

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{cases}$$

for paths $\gamma_1\colon [0,1] \to X$, $\gamma_2\colon [0,1] \to X$, if $\gamma_1(1) = \gamma_2(0)$. Clearly, it is a well-defined continuous map $[0,1] \to X$. It is called the *composition* of γ_1 and γ_2. (Do not confuse with composition of maps, i.e., applying maps successively. For paths it would make no sense, though.)

Remark: composition of paths is not associative. However, it will become associative if paths are considered up to a change of parametrization (paths γ and $\gamma \circ \varphi$ where $\varphi\colon [0,1] \to [0,1]$ is a homeomorphism identified). It will also be associative if arbitrary segments $[a,b] \subset \mathbb{R}$ will be allowed (in this case, the composition of paths should be slightly redefined). Associativity (as well as the existence of 'unities', one for each point of X and the inverses, w.r.t. the composition of paths) will definitely hold if one passes from individual paths to the so-called homotopy classes of paths, i.e., consider paths with fixed endpoints up to continuous deformations. This will feature an algebraic structure associated with the space X called the *fundamental groupoid* of X. (A *groupoid*, by definition, is a category where all arrows are invertible. Objects for the fundamental groupoid are points of X, arrows, homotopy classes of paths.)

Definition 3.6. Points x_0 and x_1 in X *can be joined by a path* (or simply *can be joined*) if there is a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$ and $\gamma_1(x_1)$.

One can show that to 'can be joined by a path' is an equivalence relation (to prove reflexivity, use constant paths; for symmetry, consider the 'inverse' path $\bar{\gamma}$ where $\bar{\gamma}(t) = \gamma(1-t)$; for transitivity, apply composition of paths).

Definition 3.7. A topological space X is *path-connected* if any two points in X can be joined by a path.

Theorem 3.19. *If X is path-connected, then X is connected.*

Proof. Let X be a path-connected topological space. Suppose it is disconnected, i.e., there are nonempty open sets U, V such that $X = U \cup V$ and $U \cap V = \varnothing$. Take a point $a \in U$ and a point $b \in V$. Consider $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$. These are nonempty (since $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$), open (as preimages of open sets under a continuous map) sets in $[0,1]$. Moreover, $\gamma^{-1}(U) \cup \gamma^{-1}(V) = [0,1]$ and $\gamma^{-1}(U) \cap \gamma^{-1}(V) = \varnothing$. Contradiction with the connectedness of $[0,1]$. Hence, X is connected. □

Remark: the converse is not true in general.

Theorem 3.20. *If X, Y are path-connected, then $X \times Y$ are connected.*

Theorem 3.21. *If X is path-connected, $f\colon X \to Y$ is continuous, then X is path-connected.*

Examples of path-connected spaces: $\mathbb{R}^n$; an open ball (convex!). S^1, S^2, S^n. Hence $\mathbb{R}P^n$, $\mathbb{C}P^n$. The square I^2, and its identification spaces. $U(n)$, $GL(n, \mathbb{C})$. $SO(n)$ (by a canonical form of an orthogonal matrix).

Theorem 3.22. *If every point of X has a path-connected neighborhood and X is connected, then X is path-connected.*

Proof. Fix a point $a \in X$ and define H as the set of all points that can be joined with a. The aim is to prove that $H = X$. Consider a point $h \in H$. There is an open neighborhood U_h of h in X which is path-connected. In particular, every point of it can be joined with h, hence with a. Hence $U_h \subset H$ for every $h \in H$. It follows that H is open. Consider $X \setminus H$. Suppose it is nonempty. Take $k \in X \setminus H$. There is an open neighborhood U_k of k in X which is path-connected. In particular, every point x of it can be joined with k. If x could also be joined with a, then k could be joined with a, which is a contradiction. Hence no point of U_k can be joined with a, i.e. $U_k \subset X \setminus H$. That means that $X \setminus H$ is open. This is a contradiction with the connectedness of X! It follows that $X \setminus H$ must be empty, i.e. $X = H$. □

Example: open sets in $\mathbb{R}^n$.

Theorem 3.23. *Every connected open subspace of a Euclidean space $U \subset \mathbb{R}^n$ is path-connected.*

Proof. Indeed, every point in an open subspace U of a Euclidean space can be surrounded by an open balled entirely contained in U. It is clearly path-connected. Then we can repeat the above proof. □

Another example: manifolds (see the next section).