§4 Topological manifolds and surface

§4.1 Surfaces

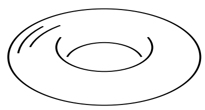
Def:

A *surface* is a two-dimensional manifold.

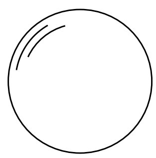
Examples:

a. $\mathbb{R}^2$

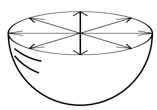
b. T^2



c. S^2



d. $\mathbb{R}P^2$



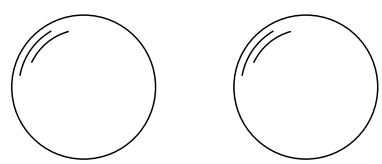
By a *closed surface* we will mean a **compact connected** surface.

Examples:

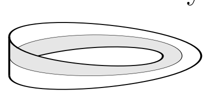
a. T^2 is a closed surface.

b. $\mathbb{R}^2$ is *not* a closed surface (non-compact).

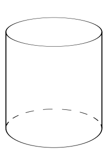
c. The union of two disjoint spheres is *not* a closed surface (disconnected).



Examples of closed surfaces can be obtained from S^2 by cutting and gluing. Ingredients: S^2, cylinders $S^1 \times [0,1]$ and Möbius strips.

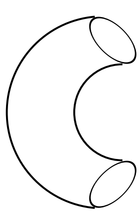


Notice that



$S^1 \times [0,1]$

$\cong$

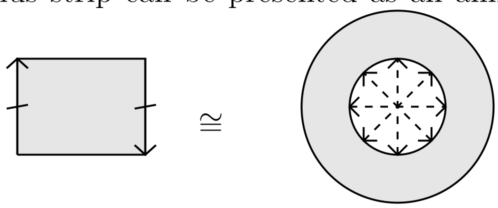


"a handle"

The boundary consists of *two* circles.

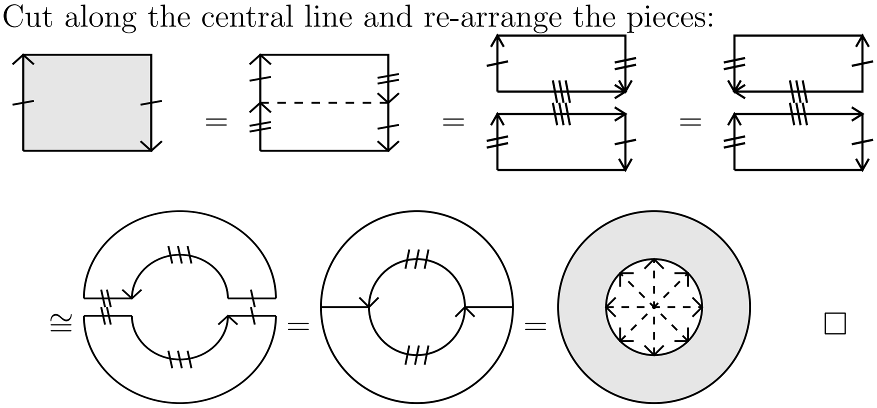
Lemma

A Möbius strip can be presented as an annulus with glued antipodal points of the inner circle:



Proof:

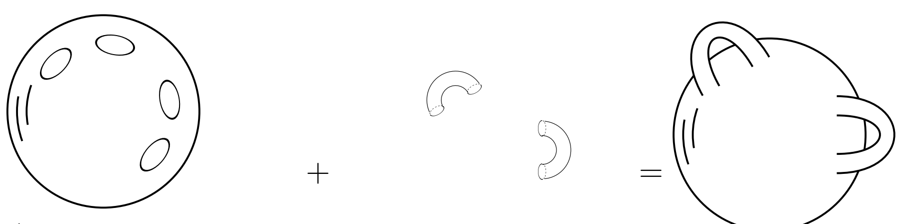
Cut along the central line and re-arrange the pieces:



Notice that, of course, the boundary of the Möbius strip is a circle. In the picture with the annulus, this is the *outer circle*.

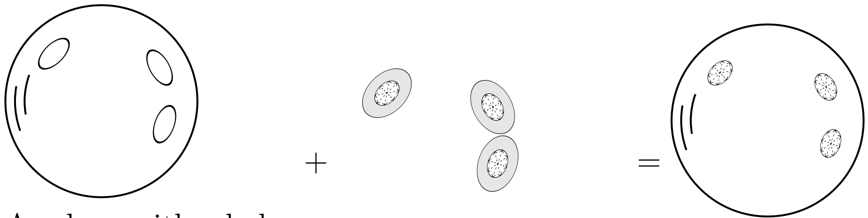
Standard surfaces

a. $H(n)$ = A sphere with n handles.



A sphere with $2n$ holes ($2n$ open disks cut out) | n cylinders (handles) | Sphere with n handles

b. $M(n)$: a sphere with n Möbius bands



A sphere with n holes (n open disks cut out)

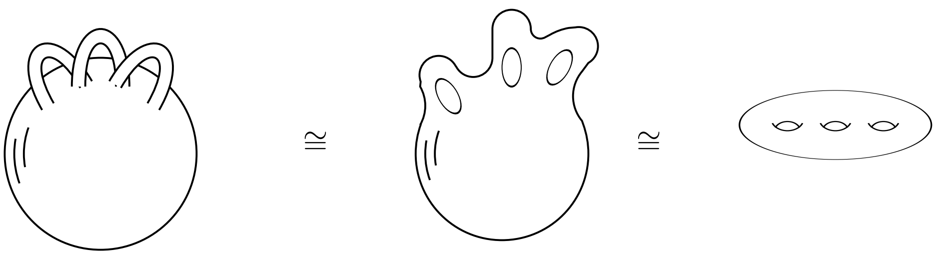
n Möbius strips

Sphere with Möbius strips (bands)

(In other words, we cut n holes and identify the antipodal points for each hole.)

Remark

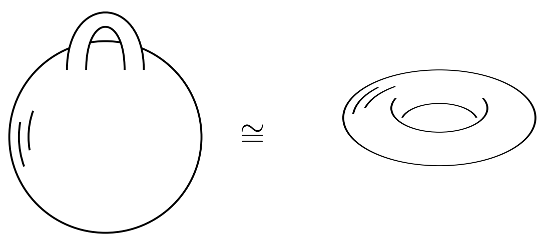
$H(n)$ is also a generalised Bagel:



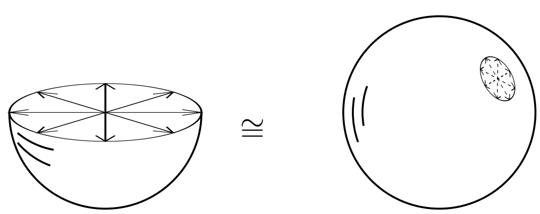
Examples:

a. $S^2 = H(0) = M(0)$

b. $T^2 \cong H(1)$:



c. $\mathbb{R}P^2 \cong M(1)$:



(see also problem sheets).

Statement

A surface is *orientable* if it does not contain a Möbius band (as a subspace) and *non-orientable* otherwise.

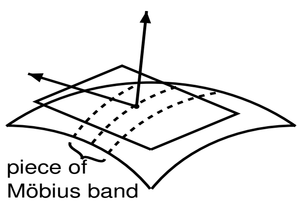
Examples:

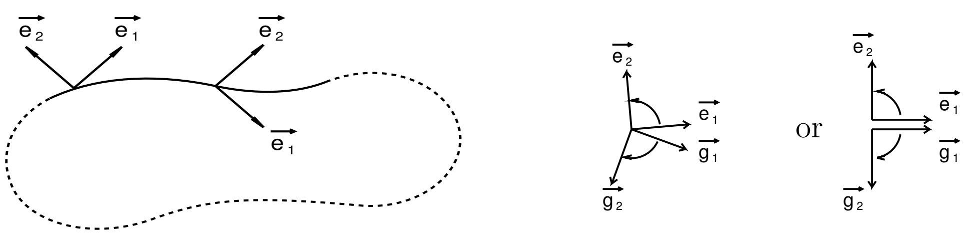
a. $\mathbb{R}P^2$ is non-orientable (see pictures above)

b. $M(n)$ is non-orientable (by the construction), for all $n > 0$.

Explanation:

Suppose a surface contains a Möbius band. Then the following is possible: (we assume for simplicity that we can embed our surface in some $\mathbb{R}^n$) - take a frame (a basis of vectors) in some tangent plane, at point P, and move it continuously along the central line of the Möbius band in our surface until we return to the starting point. One can find that the basis will *change orientation* (do an experiment with a Möbius band made of paper!):





(Here $\vec{e_1}, \vec{e_2}$ is the initial basis; $\vec{g_1}, \vec{g_2}$ is the resulting basis after we moved $\vec{e_1}, \vec{e_2}$ along this central circle back to P).

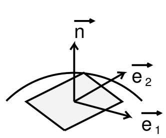
Hence *the presence of a Möbius band allows us to change the orientation of tangent frames by moving them around closed paths.*

Remark:

Orientability (and non-orientability) can be introduced for manifolds of arbitrary dimension. In the smooth case this can be done in the language of tangent frames.

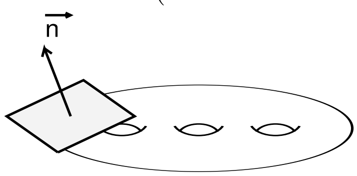
Example 1

S^2 is orientable. Indeed, $S^2 \subset \mathbb{R}^3$, one can pick a unit normal and fix the orientation in every tangent plane by requiring that $\vec{e_1} \times \vec{e_2} = \vec{n}$ (the chosen normal):



Example 2

$H(n)$ is orientable (for the same reason):



Theorem (CLASSIFICATION THEOREM FOR CLOSED SURFACES)

Every closed surface is homeomorphic either to S^2 or to $H(n), n \geq 1$, or to $M(n), n \geq 1$.

Proof is not required. (The statement *is* required!)

Hence closed surfaces are distinguished by:

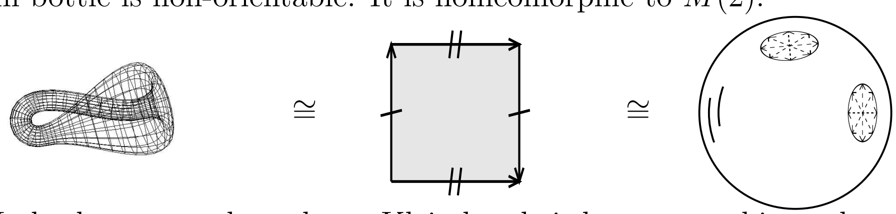
a. if it is orientable or not

b. the number n (for $H(n), M(n)$)

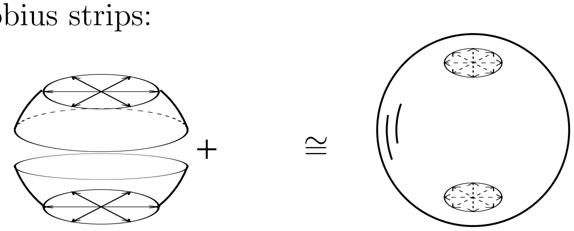
In the next section we explain the meaning of this number n (the number of handles or Möbius strips) in terms of some topological invariant ("Euler characteristic")

Example

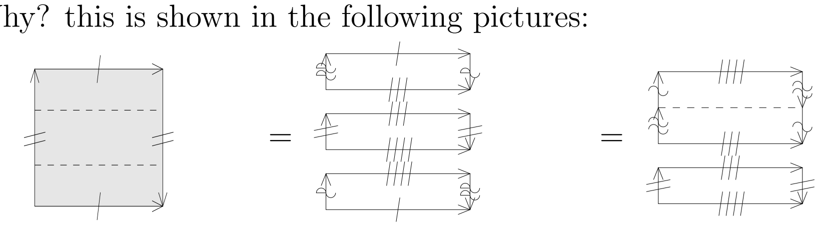
Klein bottle is non-orientable. It is homeomorphic to $M(2)$:



Indeed, one can show that a Klein bottle is homeomorphic to the result of gluing of two Möbius strips:

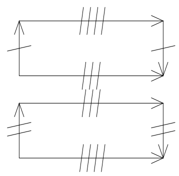


Why? this is shown in the following pictures:



(gluings are shown by various marks, like $/, //, ///, \sim$)

(we glued the bottom piece to the top piece)



Two Möbius strips
with gluing along
the boundaries

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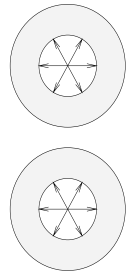

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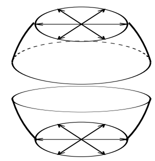

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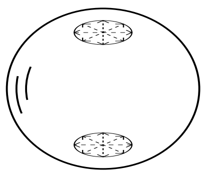

Remark

Combinations of handles and Möbius strips turns out to be equivalent to Möbius strips alone (to be elaborated later).