§5 The Euler Characteristic

§5.1 Triangulated Spaces

We are going to consider topological spaces built of blocks of a simple structure, called "simplices".

What is a simplex?

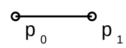
Consider $\mathbb{R}^N$ with some sufficiently large N. Let $p_0, \cdots, p_k$ be some points in $\mathbb{R}^N$.

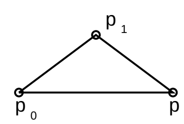
Definition

$p_0, \cdots, p_k$ are *independent* (or *in general position*) $\iff$ the vectors $\overrightarrow{p_0p_1}, \cdots, \overrightarrow{p_0p_k}$ ar linearly independent.

Examples:

1. $k = 0$: no condition 

2. $k = 1$: p_0, p_1 independent $\iff$ $p_0 \neq p_1$ 

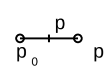
3. $k = 2$: p_0, p_1, p_2 independent $\iff$ non-collinear (not on the same line) : 

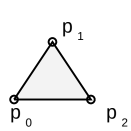
Definition:

A *simplex* in $\mathbb{R}^N$ with vertices $p_0, \cdots, p_k$ (assumed to be independent) is the subspace:

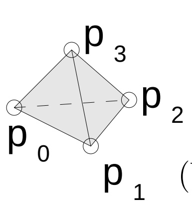
$$[p_0, \cdots, p_k] := \{p \in \mathbb{R}^N | p = \sum_{i=0}^{k} t_i p_i, t_i \geq 0, \sum_{i=0}^{k} t_i = 1\}$$

Examples:

1. $[p_0p_1] = \{t_0p_0 + t_1p_1 | t_0 + t_1 = 1, t_0, t_1 \geq 0\} = \{(1-t)p_0 + tp_1 | t \in [0, 1]\}$ 

2. $[p_0p_1p_2]$ is a triangle : 

3. $[p_0p_1p_2p_3]$ is a *tetrahedron* (3-dimensional simplex):



(Warning: this is a *solid* body, not just the surface!)

Definition:

k is called the *dimension* of $[p_0, \cdots, p_k]$.

Notation for simplices: σ, τ, π, ρ (Greek letters).

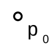
Definition:

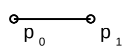
Let K be a finite collection of simplices of various dimensions in some $\mathbb{R}^N$. K is a *simplicial complex* $\iff$

1. If $\rho \in K$, then all faces of ρ also belong to K,
2. If $\rho \in K, \tau \in K$, then either $\rho \cap \tau = \oslash$ or $\rho \cap \tau$ is a common face.

What is a "face"? A *face* is obtained from a simplex $[p_0, \cdots, p_k]$ by setting some of the parameters $t_0, \cdots, t_k$ to 0. (Maybe none - then we get the whole $[p_0, \cdots, p_k]$ as its own face).

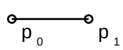
Examples:

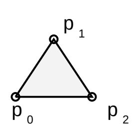
1. For a 0-simplex p_0 we have just one 0-face: p_0 itself. 

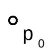
2. For a 1-simplex $[p_0 p_1]$: 

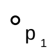
   0-faces: 

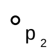
, 

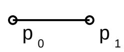
   1-face: 

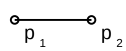
3. For a 2-simplex $[p_0 p_1 p_2]$: 

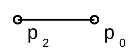
   0-faces: 

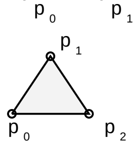
, 

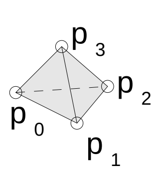
, 

   1-faces: 

, 

, 

   2-face: 

4. For a tetrahedron: 

   0-faces: p_0, p_1, p_2, p_3

   1-faces: $[p_0 p_1], [p_0 p_2], [p_0 p_3], [p_1 p_2], [p_1 p_3], [p_2 p_3]$

   2-faces: $[p_0 p_1 p_2], [p_0 p_1 p_3], [p_0 p_2 p_3], [p_1 p_2 p_3]$

   3-face: $[p_0 p_1 p_2 p_3]$

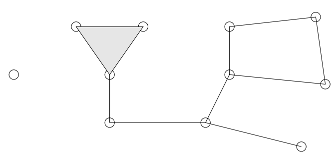
For a simplicial complex K in $\mathbb{R}^N$, the union of all its simplices is a topological space (subspace of $\mathbb{R}^N$) called the *body* of K. Notation:

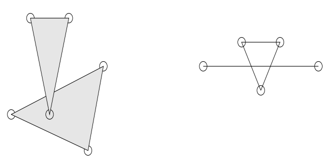
$$|K| = \bigcup_{\sigma \in K} \sigma$$

Remark

K is a collection of simplices (a set of simplices); $|K|$ is a topological space (a union of simplices).

Examples

1.  - This is a simplicial complex (in $\mathbb{R}^2$).

2.  - These are *NOT* simplicial complexes (bad intersections)

(The pictures show some simplices in the plane; would-to-be simplicial complexes are their collections; the condition (1) is assumed to hold.)

Definition

Let X be a topological space. A *triangulation* of X is a homeomorphism $\phi : X \xrightarrow{\cong} |K|$, for some simplicial complex K.

Remark

If a space X admits a triangulation, then X is compact. Indeed, for every simplicial complex K, if there is a *finite* number of simplices then the body $|K|$ is a bounded and closed (each simplex is closed) subspace of $\mathbb{R}^N$.

(To accomodate noncompact spaces, one should consider "infinite simplicial complexes" and "infinite triangulations". We won't do this.)

§5.2 Euler Characteristic

Suppose K is a simplicial complex. Notation:

$$c_k(K) = \#\{\text{simplices } \sigma \in K \,|\, \dim \sigma = k\}$$

(the number of simplices of dimension k.)

Definition

The alternating sum

$$\chi(K) = \sum c_k(K)(-1)^k = c_0(K) - c_1(K) + \cdots$$

is called the *Euler characteristic* of K. (Sum is finite).

Definition

If X is a triangulated space, i.e. there is a triangulation $X \cong |K|$ for some K, we set

$$\chi(X) := \chi(K)$$

and call $\chi(X)$ the *Euler characteristic* of the topological space X.

What if different triangulations are taken?

Theorem

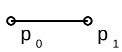
If for simplicial complexes K and L their bodies $|K|$ and $|L|$ are homeomorphic, then $\chi(K) = \chi(L)$.

(Topological invariance of the Euler characteristic.)

(Proof omitted!) □

Examples

1. $X = \{x_0\}$ (one-point space). Then $K = \{[p0]\}, c_0(K) = 1, \chi(X) = 1$.
2. *A segment* (closed):



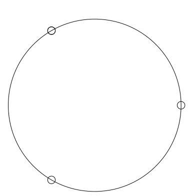
$c_0 = 2, c_1 = 1 \Rightarrow \chi = 2 - 1 = 1.$

or

$c_0 = 7, c_1 = 6 \Rightarrow \chi = 7 - 6 = 1.$

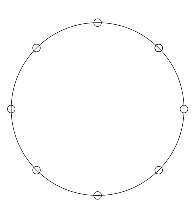
$\boxed{\chi([a, b]) = 1}$

3. *A circle*:



$c_0 = 3, c_1 = 3 \Rightarrow \chi = 3 - 3 = 0.$

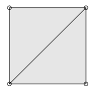
or



$c_0 = 8, c_1 = 8 \Rightarrow \chi = 8 - 8 = 0.$

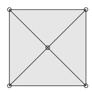
$\boxed{\chi(S^1) = 0}$

4. *A closed disk* $\overline{D^2} \cong I^2$ (a square)



$c_0 = 4, c_1 = 5, c_2 = 2 \Rightarrow \chi = 4 - 5 + 2 = 1$

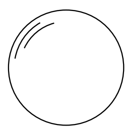
or



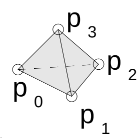
$c_0 = 5, c_1 = 8, c_2 = 4 \Rightarrow \chi = 5 - 8 + 4 = 1.$

$\boxed{\chi(\overline{D^2}) = 1}$

5. *A Sphere* S^2:



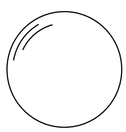
$\cong$



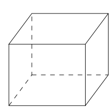
(surface of 3-simplex)

$$\left.\begin{aligned} c_0 &= 4 \\ c_1 &= 6 \\ c_2 &= 4 \end{aligned}\right\} \Rightarrow \chi = 4 - 6 + 6 = 2$$

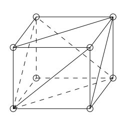
or



$\cong$



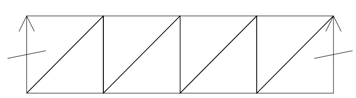
$\cong$



$$\left.\begin{aligned} c_0 &= 8 \\ c_1 &= 18 \\ c_2 &= 12 \end{aligned}\right\} \Rightarrow \chi = 8 - 18 + 12 = 2$$

$\boxed{\chi(S^2) = 2}$

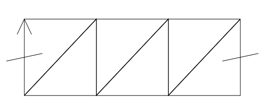
6. *A Cylinder* $S^1 \times [0, 1]$:



$$\left.\begin{aligned} c_0 &= 8 \\ c_1 &= 16 \\ c_2 &= 8 \end{aligned}\right\} \Rightarrow \chi = 8 - 16 + 8 = 0$$

$\boxed{\chi(\mathrm{Cyl}) = 0}$

7. *A Möbius strip*



$$\left.\begin{aligned} c_0 &= 6 \\ c_1 &= 12 \\ c_2 &= 6 \end{aligned}\right\} \Rightarrow \chi = 6 - 12 + 6 = 0$$

$\boxed{\chi(\text{Möb}) = 0}$

Remark

It is not a coincidence that $\chi(\text{cyl}) = \chi(\text{Möb}) = 0$. First, Cyl$= S^1 \times [0, 1]$. There is a theorem that $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$. Hence $\chi(S^1 \times [0, 1]) = 0$ because $\chi(S^1) = 0$. Next, although a Möbius strip is *not* a product (like $S^1 \times [0, 1]$), it is a "twisted product" of S^1 and $[0, 1]$ in a certain precise sense. There is a map Möb$- > S^1$ (projection onto the central circle), and locally the part that projects onto a small arc looks like a Cartesian product with $[0, 1]$. It turns out that the Euler characteristic is multiplicative with respect to such "twisted products" as well. □.

The following theorem gives a powerful tool for calculating χ:

Theorem

(The Excision Formula)

Suppose a simplicial complex is the union of two "subcomplexes" (subsets each of which is a simplicial complex) $K \cup L$. Then $K \cap L$ is also a simplicial complex, and the following formula holds:

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L)$$

Proof

Consider $c_k(K), c_k(L), c_k(K \cup L)$. Each $k-$simplex in $K \cup L$ belongs to K or L or both. Hence $c_k(K \cup L) = c_k(K) + c_k(L) - c_k(K \cap L)$, where we subtract the number of simplices counted twice in both K and L. The formula for χ follows. □

§5.3 Application to surfaces

Problem

Find χ for S^2 with n holes (i.e. $S^2 \backslash (n \text{ disks})$)

Solution

A "hole" is an open disk taken away (the boundary is left, so S^2 with a hole remains compact).

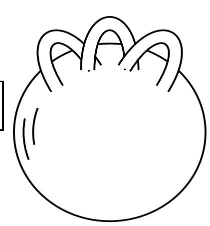
Since $D^2 \cong$ triangle, we can take a sufficiently fine triangulation of S^2 and making a hole then will mean removing the interior part of some triangle. Obviously, then $c_2 \to c_2 - 1, c_0, c_1$ remain the same. As a result

$\chi(S^2 \text{ with one hole}) = c_0 - c_1 + (c_2 - 1) = \chi(S^2) - 1$,

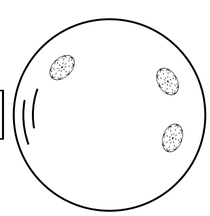
$\boxed{\chi(S^2 \text{ with } n \text{ holes}) = \chi(S^2) - n}$

It immediately follows that:

$\boxed{\chi(H(n)) = 2 - 2n}$



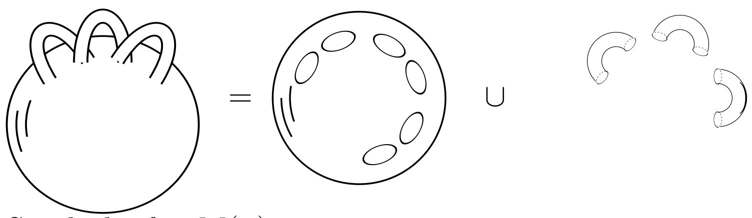
$\boxed{\chi(M(n)) = 2 - n}$



Indeed, we can apply the Excision Formula:

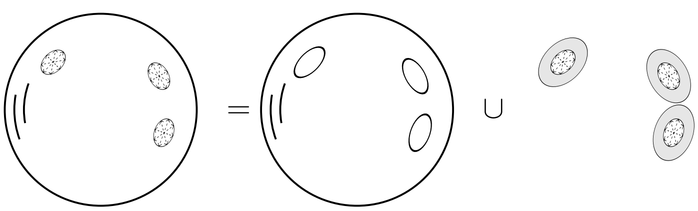
$$\begin{aligned}\chi(H(n)) &= \chi(S^2 \text{ with } 2n \text{ holes}) + \chi(n \text{ cylinders}) - \chi(2n \text{ circles})\\ &= 2 - 2n + n\cdot 0 - 2n \cdot 0\\ &= 2-2n\end{aligned}$$

(Here $H(n) = (S^2 \text{ with } 2n \text{ holes}) \cup (n \text{ “handles”=cylinders})$, and the intersection consists of $2n$ boundary circles:



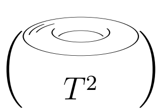
Similarly, for $M(n)$:

$$\begin{aligned}\chi(M(n)) &= \chi(S^2 \text{ with } n \text{ holes}) + \chi(n \text{ Möbius bands}) - \chi(n \text{ circles})\\ &= 2 - n + n\cdot 0 - n \cdot 0\\ &= 2-n\end{aligned}$$



Corollaries

1. $\chi(T^2) = 0$:

$\chi\Big($  $\Big) = \chi(H(1)) = 2 - 2\cdot 1 = 0$

2.

$$\chi(\mathbb{R}P^2) = \chi(M(1)) = 2 - 1 = 1$$

3.

$$\chi(\text{Klein bottle}) = \chi(M(2)) = 2 - 2 = 0$$

Remark

We see that the number “n” in $H(n)$ or $M(n)$ is encoded in the Euler characteristic of a surface. Hence, with the Classification Theorem at hand), a closed surface is classified completely by

1. orientability/non-orientability, and
2. Euler characteristic.

Example

A closed surface is obtained by cutting a hole in T^2 and gluing a Möbius strip into it. To which standard surface is this homeomorphic?

Answer

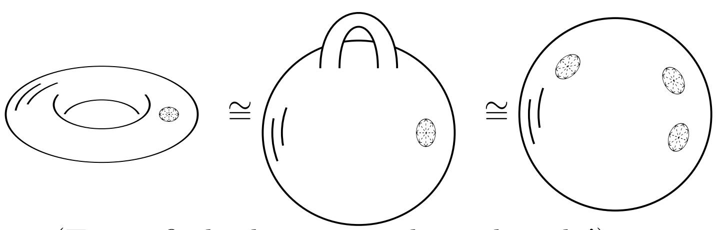
First, because there is a Möbius strip, it is *non-orientable*. Hence we look for some $M(n)$. Now, for our surface M, $\chi(M) = \chi(T^2\text{with a hole}) + \chi(\text{Möb}) - \chi(S^1)$. The last two terms are both zero, and so $\chi(M) = \chi(T^2\text{with a hole})$.

Let us now find $\chi(T^2\text{with a hole})$:

Notice that $T^2 = (T^2\text{with a hole}) \cup (\text{disk}D^2)$, with intersection S^1. This gives $0 = \chi(T^2) = \chi(T^2\text{with a hole}) + (\chi(D^2) = 1) - (\chi(S^1) = 0)$ from which we see that $\chi(T^2\text{with a hole}) = -1$ and finally $\chi(M) = -1$.

If we now compare this with $\chi(M(n)) = 2 - n$, we conclude that $n = 3$, and that $M \cong M(3)$.

Pictures



(Try to find a homeomorphism directly!)