§8 Parallel transport

8.1 Equation of parallel transport

Consider a vector bundle $E \to B$. We would like to compare vectors belonging to fibers over different points. Recall that this was the main problem preventing from the possibility of differentiating sections without an extra structure such as a connection. We wish to approach it now from a different angle.

Suppose initially that the bundle $E$ is trivial, i.e., $E = B \times \mathbb{R}^k$ or $E = B \times \mathbb{C}^k$. In this case fibers over different points can be canonically identified with the same space $\mathbb{R}^k$ or $\mathbb{C}^k$. Two vectors over different points are the ‘same’ if they are equal as elements of this standard fiber. If we are speaking about the tangent vectors (say, to an open set in $\mathbb{R}^n$), then they are visualized as directed segments attached to the corresponding points. Two vectors at points $P$ and $Q$ are equal if the directed segments specifying them can be transformed one into another by a parallel translation. Parallel translation, in particular, preserves lengths of vectors.

Now let $E$ be a subbundle of a trivial bundle $B \times V$. (For example, the tangent bundle for a surface in $\mathbb{R}^n$ or the tautological line bundle over a projective space.) We can still try to apply ‘parallel translation’ in the ambient space $V$, i.e., the identity transformation, but then a problem will be that a vector from a fiber $E_P$ over a point $P \in B$ ‘translated’ to a point $Q$, i.e., the same vector as an element of the space $V$, will almost never be in the space $E_Q$. For example, a tangent vector to the circle $x^2 + y^2 = 1$ at $P = (1, 0)$ translated identically to $Q = (0, 1)$ will be normal to the circle at that point.

A remedy that can be suggested is to consider the (identical) ‘parallel translation’ of a vector from space $E_P \subset V$ in the ambient space $V$ followed by (say, orthogonal) projection onto the space $E_Q \subset V$, thus forcefully making the result of the operation to be in the required space. However, a problem remains: thus constructed linear map $E_P \to E_Q$ ceases to preserve lengths of vectors; even worse, it can map certain non-zero vectors to zero. In the above example, so defined map from $T_{(1,0)}S^1$ to $T_{(0,1)}S^1$ will be identically zero. Such a problem does not exist if the ‘starting’ point $P$ and the ‘final’
point $Q$ are sufficiently close: in this case the projection map is close to the identity on $E_P$ and is invertible. Moreover, the lengths of vectors are preserved modulo $\varepsilon^2$ if $\varepsilon$ is the order of distance between $P$ and $Q$. (Note that $\cos \varepsilon = 1 + O(\varepsilon^2).$)

The correct understanding of a parallel translation for a subbundle of a trivial vector bundle $E \subset B \times V$ is thus as follows. To translate a vector $u \in E_P \subset V$ obtaining a vector in $E_Q \subset V$ one has to do it in steps each time moving from a point in the base to an infinitesimally close point. In other words, we need to have a curve $x(t)$ joining the points $P$ and $Q$ so that $x(0) = P$ and $x(1) = Q$. The result of the translation in general will depend on this curve (not only on endpoints). Infinitesimal translation $\tau(x + \varepsilon X, x)$ from $E_x$ to $E_{x+\varepsilon X}$ is defined by

$$\tau(x + \varepsilon X, x)u = P_{x+\varepsilon X}u.$$ 

Here we use a formal parameter $\varepsilon$ such that $\varepsilon^2 = 0$. For a point $x \in B$ and a tangent vector $X \in T_xB$, the sum $x_{\varepsilon} = x + \varepsilon X$ makes good sense as a point of $B$ that is ‘infinitesimally close’ to $x$. Denote the result of the infinitesimal parallel translation by $u_{\varepsilon}$. We have

$$u_{\varepsilon} = P_{x_{\varepsilon}}u \quad \text{or} \quad P_{x_{\varepsilon}}(u_{\varepsilon} - u) = 0$$

(because the the projector on a fiber maps every vector in the fiber to itself). Since the difference between $P_{x_{\varepsilon}}$ and $P_x$ is of order $\varepsilon$, and the same is true for $u_{\varepsilon} - u$, it is possible to replace $P_{x_{\varepsilon}}$ by $P_x$ in the last equation. We arrive at

$$P_x(u_{\varepsilon} - u) = 0$$

as the equation defining an infinitesimal parallel translation. If we are given a curve, then $x = x(t)$, $x_{\varepsilon} = x(t + \varepsilon)$, $u = u(t)$, and $u_{\varepsilon} = u(t + \varepsilon)$; the equality (1) becomes

$$P_{x(t)}(u(t + \varepsilon) - u(t)) = 0,$$

which is equivalent to

$$P_{x(t)} \left( \frac{du}{dt} \right) = 0.$$

The differential equation (2) is called the equation of parallel transport. Recalling that the ordinary derivative in the ambient space followed by the
projection is the covariant derivative on the bundle \( E \), we may rewrite the equation of parallel transport also as
\[
\frac{\nabla u}{dt} = 0, \tag{3}
\]
which makes sense for an arbitrary vector bundle endowed with a connection.

The outcome of our investigation can be summarized in the following definition.

**Definition 8.1.** The **parallel translation** (or **parallel transport**) on a vector bundle \( E \to B \) endowed with a connection is a linear transformation \( \tau(Q, P, \gamma) : E_P \to E_Q \) for each pair of points \( P, Q \in B \) joined by a curve \( \gamma : t \mapsto x(t) \), so that \( x(0) = P \) and \( x(1) = Q \), that sends a vector \( u_0 \in E_P \) to the vector \( u_1 = u(1) \in E_Q \), the value at time \( t = 1 \) of the solution \( u(t) \) of the differential equation (3) with the initial condition \( u(0) = u_0 \).

In the simplest cases it is possible to find the parallel translation without solving the differential equation.

**Example 8.1.** If a surface \( M^2 \subset \mathbb{R}^3 \) is developable on a flat surface (a piece of a plane), then the parallel translation on this surface, i.e., the parallel translation in the tangent bundle, is the same as on that flat surface.

Equation of parallel transport:
\[
\nabla_t u = 0
\]
or, using a local frame, so \( u = e_i u^i \),
\[
\frac{d u^i}{dt} + \dot{x}^a A^i_{aj}(x(t)) u^j = 0.
\]

For an infinitesimal parallel transport we thus obtain:
\[
\tau(x_0 + X, x_0) = 1 - X^a A_a(x_0),
\]
where \( \varepsilon^2 = 0 \).

Solution by multiplicative integral.

**Theorem 8.1.** For a connection compatible with metric, parallel transport is an orthogonal or unitary transformation (in the real or complex cases, respectively).
8.2 Geodesic lines

**Definition 8.2.** On a manifold $M$ with a connection $\nabla$, *geodesic lines* or simply *geodesics* are defined as curves $x = x(t)$ satisfying the equation

$$\nabla_x \dot{x} = 0,$$

(4)
i.e., the condition that the velocity vector $\dot{x}$ is parallel along the curve.

Intuitively, the condition defining geodesics means that they are the “straightest” lines on $M$. The equation of geodesics (4) in an explicit form is

$$\frac{d^2 x^a}{dt^2} + \Gamma^a_{bc} \frac{dx^b}{dt} \frac{dx^c}{dt} = 0.$$  

(5)

**Example 8.2.** On $\mathbb{R}^n$, geodesics are precisely the straight lines in an affine parametrization:

$$x^a = v^a t + x_0^a$$
in affine coordinates. Indeed, in affine coordinates, $\Gamma^a_{bc} = 0$ and the equation reduces to

$$\frac{d^2 x^a}{dt^2} = 0.$$

**Theorem 8.2.** Suppose $M \subset \mathbb{R}^N$ is a submanifold in Euclidean space $\mathbb{R}^N$. Then a curve is a geodesic on $M$ if it belongs to $M$ and its acceleration vector in the ambient space $\mathbb{R}^N$ is perpendicular to $M$.

**Proof.** Indeed, the orthogonal projection

$$P \left( \frac{d^2 x}{dt^2} \right)$$
onto $T_x M$ is precisely the covariant derivative $\nabla_x \dot{x}$.

The statement of Theorem 8.2 has a clear physical meaning: a particle moves along a geodesic on $M$ if and only if the sum of all forces acting on it has no component along $M$; there is only a ‘reaction force’ normal to $M$ and making the particle stay on $M \subset \mathbb{R}^N$. Motion along a geodesic is a free motion on $M$.

**Example 8.3.** Geodesics on $S^2$ (and on $S^n$ for any $n$) are the *great circles*, i.e., the sections of $S^2$ by planes through the center, taken with the natural parametrization (up to a constant factor).
Theorem 8.3. For a Riemannian manifold \( M \), the equation of geodesics can be written as
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0
\] (6)
where \( L \) is a function on \( TM \) given by the formula
\[
L = \frac{1}{2}(\dot{x}, \dot{x}) = \frac{1}{2}g_{ab}(x)\dot{x}^a\dot{x}^b. \tag{7}
\]

Proof. We have
\[
\frac{\partial L}{\partial \dot{x}^a} = g_{ab}(x)\dot{x}^b.
\]
Hence the time derivative will be
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^a} \right) = \frac{d}{dt} \left( g_{ab}(x)\dot{x}^b \right) = \partial_c g_{ab} \dot{x}^c \dot{x}^b + g_{ab} \ddot{x}^b = \frac{1}{2} (\partial_c g_{ab} + \partial_b g_{ac}) \dot{x}^b \dot{x}^c + g_{ab} \ddot{x}^b.
\]
We also have
\[
\frac{\partial L}{\partial x^a} = \frac{1}{2} \partial_a g_{bc} \dot{x}^b \dot{x}^c.
\]
Hence the Euler–Lagrange equation is
\[
g_{ab} \ddot{x}^b + \frac{1}{2} (\partial_b g_{ac} + \partial_c g_{ab} - \partial_a g_{bc}) \dot{x}^b \dot{x}^c = 0
\]
or, equivalently (by multiplying by \( g^{ap} \) with summation over \( a \) followed by renaming \( p \leftrightarrow a \)),
\[
\dddot{x}^a + \frac{1}{2} g^{ap}(\partial_b g_{pc} + \partial_c g_{pb} - \partial_p g_{bc}) \dot{x}^b \dot{x}^c = 0.
\]
This is exactly
\[
\dddot{x}^a + \Gamma^a_{bc} \dot{x}^b \dot{x}^c = 0
\]
by the Christoffel formulas.

Corollary 8.1. The parameter \( t \) for a geodesic on a Riemannian manifold is the natural parameter (arc length) up to a constant factor.

Proof. For any equations of the form (7) if a function \( L \) does not explicitly depend on \( t \) there is the ‘conservation of energy’: the function
\[
E = \frac{\partial L}{\partial \dot{x}} \dot{x} - L,
\]
is constant along solutions of (7) (check!). In our case the energy $E$ is the ‘kinetic energy’ and coincides with $L$:

$$E = \frac{\partial L}{\partial \dot{x}^a} \dot{x}^a - L = g_{ab}(x)\dot{x}^b \dot{x}^a - \frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b = \frac{1}{2} g_{ab}(x) \dot{x}^a \dot{x}^b = L.$$  

Therefore $(x, \dot{x})$ is constant, i.e., the magnitude of the velocity vector $\dot{x}$ is constant. Since for the arc length we have

$$s = \int_{t_0}^{t} |\dot{x}| dt ,$$

we conclude that $s = ct + t_0$. Alternatively, the same can be seen directly from the definition of geodesics:

$$\frac{d(\dot{x}, \dot{x})}{dt} = \partial_x (\ddot{x}, \dot{x}) = (\nabla_x \dot{x}, \dot{x}) + (\dot{x}, \nabla_x \dot{x}) = 2(\nabla_x \dot{x}, \dot{x}) = 0$$

due to the equation of geodesics $\nabla_x \dot{x} = 0$. \(\square\)

One can use equation (6) as a shortcut for calculating Christoffel symbols.

**Example 8.4.** For the metric of the sphere $S^2$ written in coordinates $\theta, \varphi$ we have

$$ds^2 = R^2 (d\theta^2 + \sin^2 \theta d\varphi^2).$$

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**Remark 8.1.** Equation (6) is the necessary and sufficient condition of extremum for the functional

$$S[\gamma] = \int_{t_0}^{t_1} L(x(t), \dot{x}(t)) dt$$

defined by a given function $L \in C^\infty(TM)$, which is called a Lagrangian. Equation (6) is called the Euler–Lagrange equation with the Lagrangian $L$.

**Remark 8.2.** We have defined geodesics as the **straightest** lines on a manifold. It turns out that on a Riemannian manifold they are also the **shortest**. They satisfy the Euler–Lagrange equation not only for the Lagrangian

$$L = \frac{1}{2}(\dot{x}, \dot{x}),$$

6
but also for another Lagrangian

\[ L' = |\dot{x}| = \sqrt{2L}. \]

The corresponding functional is nothing but the arc length of a curve between given \( t_0 \) and \( t_1 \). Thus geodesics realize the extrema of length. Moreover, it can be shown, that for sufficiently close points on a geodesic, the corresponding geodesic segment gives the minimum of length.

**Example 8.5.** For two points on the sphere, in the generic case there is one geodesic giving the minimum of length and another, the maximum.