§9 Curvature and parallel transport

9.1 Influence of curvature on parallel transport

Suppose we have a vector bundle $E \to B$ endowed with a connection.

Consider points $x_0$ and $x_1$ in $B$. The parallel transport along a curve $\gamma: x = x(t)$ joining $x_0$ and $x_1$ is an isomorphism $E_{x_0} \to E_{x_1}$. Question: does it depend on a choice of $\gamma$?

The situation may be compared to that for line integrals. As we know, for fixed endpoints, a line integral $\int \omega$ is independent on path if the form $\omega$ is exact and is independent on path in the same homotopy class if the form $\omega$ is closed. Equivalently, one may analyze integrals over closed contours. An integral $\oint \omega$ over such a contour is zero for all exact 1-forms $\omega$ and also for all closed 1-forms if the contour is the boundary of a 2-surface.

For connections, the analog of being ‘exact’ is the triviality of a connection: by a choice of local frames all local connection 1-forms can be made zero and the transition functions can be made identity. In particular, that means that our vector bundle is trivial. Parallel transport in this case does not depend on path.

The analog of being ‘closed’ is the flatness of a connection, i.e., the vanishing of the curvature. One can show that in such case by a choice of local frames all local connection 1-forms can be made zero and the transition functions can be made local constants (although not necessarily identity). One can also show that parallel transport for a flat connection does not depend on path in a given homotopy class. (For non-homotopic paths parallel transports may be different.)

Now let us investigate this in a greater detail.

Recall the notion of homotopy. Consider two paths $\gamma_0: [0, 1] \to M$ and $\gamma_1: [0, 1] \to M$ with the same endpoints: $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$. They are said to be homotopic (with fixed endpoints) if there is a map $\Gamma: [0, 1]^2 \to M$ such that $\Gamma(0, t) = \gamma_0(t)$ and $\Gamma(1, t) = \gamma_1(t)$ for all $t \in [0, 1]$.

**Theorem 9.1.** If the curvature of a connection on $E \to B$ vanishes, then the parallel transport does not depend on a path in a given homotopy class (with fixed endpoints). And the converse is true.
Proof. Consider two homotopic paths with fixed endpoints. We may consider them as a family of paths $\gamma_s$ with the same endpoints. Consider the parallel transport $\tau_s(t) := \tau(\gamma_s(t), P, \gamma_s): E_P \to E_{\gamma_s(t)}$. Here we denote the endpoints as $P = \gamma_s(0)$ and $Q = \gamma_s(1)$. We are interested in the transformations $\tau_s = \tau_s(1): E_P \to E_Q$. Take some initial value $u_0 \in E_P$ and consider $u(s, t) := \tau_s(t)u_0$. It satisfies

$$\nabla_t u(s, t) = 0, \quad u(s, 0) = u_0.$$ 

Introduce $w(s, t) = \nabla_s u(s, t)$. In particular we have

$$w(s, 0) = \frac{\partial}{\partial s} u(s, 0) = 0 \quad \text{and} \quad w(s, 1) = \frac{\partial}{\partial s} u(s, 1) = \frac{\partial}{\partial s} \tau_s u_0.$$ 

From the differential equation for $u(s, t)$ we obtain the equation for $w(s, t)$:

$$0 = \nabla_s \nabla_t u(s, t) = \nabla_t \nabla_s u(s, t) - F(\partial_t, \partial_s)u(s, t)$$

or:

$$\nabla_t w(s, t) = F_{ts} u(s, t)$$

together with the initial condition $w(s, 0) = 0$. If $F = 0$, then $w(s, t)$ satisfies a linear homogeneous equation with zero initial condition, therefore it is zero. We conclude that $\tau_s$ does not depend on $s$ as claimed. (A proof of the converse statement is left to the reader.)

To see the effect of curvature in greater detail let us consider the infinitesimal case. Let us define an ‘infinitesimal closed contour’. Fix a point $x_0 \in M$ and take two tangent vectors $X, Y \in T_{x_0}M$. Consider a 2-dimensional surface $\Gamma$ passing through $x_0$ so that the tangent plane to $\Gamma$ at $x_0$ is spanned by $X, Y$. We may assume that $D$ is parametrized by two parameters $t, s$ so that $X = \partial_t$ and $Y = \partial_s$ at $x_0$. By the infinitesimal contour spanned by $X, Y$ we shall mean the ‘coordinate rectangle’ in $D$ made by the coordinate lines of $t$ and $s$ where $0 \leq t \leq \varepsilon$ and $0 \leq s \leq \eta$. We shall perform calculations neglecting $\varepsilon^2$ and $\eta^2$ and all higher terms, but not $\varepsilon\eta$.

Theorem 9.2. The parallel transport $\tau: E_{x_0} \to E_{x_0}$ along the infinitesimal closed contour spanned by vectors $X, Y \in T_{x_0}M$ is given by the value of the curvature 2-form on $X$ and $Y$:

$$\tau = 1 - 2\varepsilon\eta F(X, Y).$$

(1)
Proof. Let $A = A_a(x)dx^a$ be the local connection 1-form. We shall denote its restriction to $\Gamma$ by the same letter so on $\Gamma$ we have $A = A_t dt + A_s ds$. Denote the vertices of our contour by 1, 2, 3, 4 so that the coordinates of 1 = $x_0$ are (0, 0), the coordinates of 2 are $(\varepsilon, 0)$, of 3, $(\varepsilon, \eta)$, and of 4, (0, $\eta$). Thus for parallel transports along the edges we obtain:

$$
\begin{align*}
\tau(2, 1) &= 1 - \varepsilon A_t(0, 0), \\
\tau(3, 2) &= 1 - \eta A_s(\varepsilon, 0), \\
\tau(4, 3) &= 1 + \varepsilon A_t(\varepsilon, 0), \\
\tau(1, 4) &= 1 + \eta A_s(0, \eta).
\end{align*}
$$

Therefore

$$
\tau = \tau(1, 4)\tau(4, 3)\tau(3, 2)\tau(2, 1) =
(1 + \eta A_s(0, \eta))(1 + \varepsilon A_t(\varepsilon, 0))(1 - \eta A_s(\varepsilon, 0))(1 - \varepsilon A_t(0, 0)) =
(1 + \varepsilon A_t(\varepsilon, \eta) + \eta A_s(0, \eta) + \varepsilon \eta A_s(0, \eta)A_t(\varepsilon, \eta))
(1 - \varepsilon A_t(0, 0) - \eta A_s(\varepsilon, 0) + \varepsilon \eta A_s(0, 0)A_t(0, 0)) =
(1 + \varepsilon A_t(0, \eta) + \eta A_s(0, 0) + \varepsilon \eta A_s(0, 0)A_t(0, 0))
(1 - \varepsilon A_t(0, 0) - \eta A_s(\varepsilon, 0) + \varepsilon \eta A_s(0, 0)A_t(0, 0)) =
1 - \varepsilon A_t - \eta A_s(\varepsilon, 0) + \varepsilon A_t(0, \eta) + \eta A_s + \varepsilon \eta (A_s A_t - A_t A_s - A_t A_t + A_s A_t) =
1 - \varepsilon \eta (\partial_t A_s - \partial_s A_t + [A_t, A_s]) = 1 - \varepsilon \eta F_{ts}.
$$

Here we suppress the arguments (0, 0) and use $F_{ts}$ for the component of the restriction of the curvature 2-form $F$ on the surface $\Gamma$. For this restriction we have

$$
F = F_{ts} dt \wedge ds.
$$

Since

$$
F(X, Y) = F(\partial_t, \partial_s) = \frac{1}{2} F_{ts},
$$

we arrive at (1).

To obtain a global form of the relation between curvature and parallel transport along a closed contour, let us restrict ourselves to the commutative case $^1$.

$^1$The general case would require a generalization of the Stokes formula to product integrals.
Consider a two-dimensional real vector bundle $E \rightarrow B$ with a metric and a connection compatible with metric. Let a closed contour $\gamma$ be the boundary of a 2-chain $D$. Fix a starting point $x_0$ on $\gamma$. Denote the the parallel transport $E_{x_0} \rightarrow E_{x_0}$ by $\tau(x_0, \gamma)$. Since parallel transport respects metric, $\tau(x_0, \gamma)$ is an orthogonal transformation. The curvature 2-form takes values in antisymmetric operators, which commute. Let us assume for simplicity that the chain $D$ is contained entirely in a domain where we can introduce a local trivialization for our bundle. Using an orthonormal frame we can write the curvature 2-form for $E$ as

$$F = \begin{pmatrix} 0 & \mathcal{F} \\ -\mathcal{F} & 0 \end{pmatrix}$$

where $\mathcal{F}$ is an ordinary (scalar-valued) 2-form on $B$.

**Theorem 9.3.** For a two-dimensional real vector bundle with a metric connection, the parallel transport $\tau(x_0, \gamma)$ along a closed contour $\gamma = \partial D$ is the rotation through the angle

$$\Delta \varphi = \int_D \mathcal{F}.$$  

(In particular, it does not depend on the starting point $x_0$.)

**Proof.** Consider a local orthonormal frame $e_1, e_2$. Then for the matrix-valued connection 1-form $A$ we have

$$A = \begin{pmatrix} 0 & A_a dx^a \\ -A_a dx^a & 0 \end{pmatrix}$$

where $A = A_a dx^a$ is an ordinary 1-form. Due to commutativity, the solution of the equation of parallel transport can be written as

$$\exp \left( -\int_{t_0}^{t_1} A \right) = \exp \begin{pmatrix} 0 & -\int_{t_0}^{t_1} A \\ \int_{t_0}^{t_1} A & 0 \end{pmatrix}$$

For the closed contour $\gamma = \partial D$ we have

$$\tau = \exp \left( \int_\gamma A \right) = \exp \left( \int_D dA \right) = \exp \int_D F$$

by the Stokes formula, since

$$F = \begin{pmatrix} 0 & \mathcal{F} \\ -\mathcal{F} & 0 \end{pmatrix} = dA = \begin{pmatrix} 0 & dA \\ -dA & 0 \end{pmatrix}$$
In particular we see that the parallel transport along $\gamma$ is the rotation of the space $E_{x_0}$ through the angle

$$\Delta \varphi = \int_D \mathcal{F}$$

(which does not depend on $x_0$).

**Corollary 9.1.** On a surface $M^2 \subset \mathbb{R}^3$ or on any 2-dimensional Riemannian manifold $M^2$, the parallel transport along a closed contour $\gamma = \partial D$ is the rotation through the angle $\Delta \varphi$ where

$$\Delta \varphi = \int_D K dS.$$ 

Here $K$ is the Gaussian curvature and $dS = \sqrt{g} du \wedge dv$ is the area element.

**Proof.** On a manifold $M^2$, the curvature 2-form in an orthonormal frame can be written as

$$\Omega = \begin{pmatrix} 0 & \Omega_{12} \\ -\Omega_{12} & 0 \end{pmatrix}$$

where

$$\Omega_{12} = K e^1 \wedge e^2.$$ 

Here $K$ is the Gaussian curvature and $e^1, e^2$ is a dual orthonormal basis of covectors. In other words,

$$\Omega = \begin{pmatrix} 0 & K dS \\ -K dS & 0 \end{pmatrix}$$

and we can use the previous statement. \qed

### 9.2 Gauß–Bonnet Theorem

On a 2-dimensional Riemannian manifold $M^2$ (for example, a surface in $\mathbb{R}^3$), consider a triangle $\triangle ABC$ whose sides are geodesic segments. Denote by $\hat{A}$, $\hat{B}$ and $\hat{C}$ the angles at the respective vertices.

**Theorem 9.4.** For the sum of angles of a geodesic triangle $\triangle ABC$ the following formula holds:

$$\hat{A} + \hat{B} + \hat{C} = \pi + \int_{\triangle ABC} K dS.$$
Proof. Consider a geodesic triangle $\triangle ABC$. The method of a proof is to consider the parallel transport of a given vector $X$ along the boundary of $\triangle ABC$ and to use the formula for the angle of rotation given above. As $X$ we shall take a vector at $A$ tangent to the side $AB$. Introduce the following notation. Denote the angles as $\alpha = \hat{A}$, $\beta = \hat{B}$, and $\gamma = \hat{C}$. Consider vectors such as $\overrightarrow{AB} \in T_A M$, $\overrightarrow{BC} \in T_B M$, etc., tangent to the respective sides and pointing in the corresponding directions. Their magnitudes are not important, but for certainty let us assume that they represent initial velocities for the respective geodesics such that the final points can be achieved in unit time. Denote parallel transports along the sides as $\tau(B, A): T_A M \to T_B M$, etc. Let $X = \overrightarrow{AB}$ and consider its parallel transports along the sides:

$$X_B := \tau(B, A)X, \quad X_C := \tau(C, B)X_B, \quad X' := \tau(A, C)X_C.$$ 

We have the following relations for the angles (we systematically rely on the fact that the parallel transport of a vector tangent to a geodesic along this geodesic will remain tangent to it):

$$(\overrightarrow{X_B}, \overrightarrow{BC}) = \pi - \beta$$

$$(\overrightarrow{X_C}, -\overrightarrow{CB}) = \pi - \beta, \quad (\overrightarrow{X_C}, -\overrightarrow{AC}) = \pi - \beta - \gamma$$

$$(\overrightarrow{X'}, \overrightarrow{AC}) = \pi - \beta - \gamma,$$

on the other hand,

$$(\overrightarrow{X'}, \overrightarrow{AC}) = (\overrightarrow{X'}, \overrightarrow{X}) + (\overrightarrow{X}, \overrightarrow{AC}) = -\Delta \varphi + \alpha.$$ 

Therefore

$$\pi - \alpha - \beta - \gamma = -\Delta \varphi.$$ 

Applying the formula for $\Delta \varphi$ completes the proof. 

The quantity $\pi - (\hat{A} + \hat{B} + \hat{C})$ is known as the defect of a triangle. The negative of the defect is called, the excess. For a flat surface the defect (or excess) is zero. Theorem 9.4 says that the excess of a triangle is precisely the integral of the Gaussian curvature.

Example 9.1. For a sphere $S^2$ of radius $R$, the Gaussian curvature is $R^{-2}$ and the excess is the corresponding solid angle (the area divided by $R^2$). For
example, for the triangle $ABC$ where the vertices have $\theta, \varphi$ coordinates equal to $(\frac{\pi}{2}, 0), (\frac{\pi}{2}, \frac{\pi}{2})$, and $(0, \frac{\pi}{2})$, respectively, the sum of the angles is
\[
\frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} = \frac{3\pi}{2}
\]
and the excess is $\frac{\pi}{2}$. This is exactly one-eighth of the area of the sphere divided by $R^2$. (For a unit sphere, the excess is exactly the area of the triangle.)

**Example 9.2.** For the Lobachevski plane, the excess is negative and is, up to a constant, the negative of the area of the triangle (i.e., the opposite to the case of the sphere).

Consider a compact oriented manifold $M$ of dimension 2. An example is surface in $\mathbb{R}^3$ such as a sphere or a torus. Consider a triangulation of $M$. (It is not hard to construct a triangulation for a particular surface in $\mathbb{R}^3$; in fact, all compact oriented 2-manifolds are surfaces, see below.) We can assume that the edges of the triangulation are segments of geodesics. (It is intuitively clear that for a sufficiently fine triangulation, so that all vertices are close, curves representing edges can be made “as straight as possible”, i.e., made geodesics.)

**Theorem 9.5 (Gauß–Bonnet Theorem).** For an arbitrary Riemannian metric on a compact oriented 2-dimensional manifold $M^2$,
\[
\frac{1}{2\pi} \oint_{M^2} K dS = \chi(M^2),
\]
where $K$ is the Gaussian curvature.

**Proof.** Consider the formula for the sum of angles of a geodesic triangle $\triangle ABC$
\[
\hat{A} + \hat{B} + \hat{C} = \pi + \oint_{\triangle ABC} K dS
\]
and take the sum over all triangles of the triangulation. Let $c_0$ denote the number of vertices; $c_1$, of edges; $c_2$, of triangles. We arrive at:
\[
\sum \text{all angles} = \pi \cdot c_2 + \oint_M K dS.
\]
Note that the sum of all angles equals $2\pi$ (total angle at a vertex) multiplied by the number of vertices $c_0$. We have

$$2\pi \cdot c_0 - \pi \cdot c_2 = \int_M K \, dS.$$ (3)

To transform the LHS, let us introduce the number of edges into the formula. Since in each triangle there are three edges and in the triangulation of $M$ each edge appears as a common side of two triangles (due to the fact that $M$ is a manifold without boundary), we have a relation

$$3c_2 - 2c_1 = 0.$$ (4)

Multiplying it by $\pi$ and adding to (3), we obtain

$$2\pi \cdot c_0 - 2\pi \cdot c_1 + 2\pi \cdot c_2 = \int_M K \, dS,$$ (5)

which gives the desired formula (2).

Example 9.3. The sphere $S^2$ cannot have a Euclidean metric, i.e., the one that can be written as $ds^2 = (dx^1)^2 + (dx^2)^2$ in all charts of some atlas. Indeed, if it had one, then $K$ for this metric would be zero, therefore, $\int_{S^2} K \, dS$ would be zero, but it equals $2\pi \chi(S^2) = 4\pi \neq 0$.

Notice that the standard metric of the sphere obtained from the standard embedding into $\mathbb{R}^3$ is not Euclidean. The point, however, is that it is impossible in principle to endow $S^2$ by a Euclidean metric. Note also that it is a global statement: clearly, on a piece of the sphere, which is the same as a piece of a plane, it is always possible to have a Euclidean metric.

Remark 9.1. In equation (2), the LHS is defined in terms of a Riemannian metric while the RHS, in terms of a triangulation. The equality of the two quantities implies that the integral in the LHS is independent of a metric and that the RHS (the Euler characteristic) is independent of a triangulation.

The following statement should be regarded as a part of general mathematical culture.

Theorem 9.6 (“Classification Theorem for Closed Surfaces”). Any compact orientable 2-dimensional manifold $M$ is a ‘sphere with $g$ handles’, where the number $g = 0, 1, 2, \ldots$ is called the genus of $M$. 

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‘Is’ in the statement of the theorem means: ‘is diffeomorphic to’ or ‘is homeomorphic to’ (this is the same for 2-manifolds).

Genus is the only topological invariant of a compact orientable 2-manifold.

Here a sphere with handles is obtained from \( S^2 \subset \mathbb{R}^3 \) by cutting 2g holes (small enough to be disjoint) and gluing to the resulting manifold with boundary (the boundary consists of 2g circles) g ‘handles’, which are cylinders \( S^1 \times [0, 1] \). One can show that the result is a compact 2-manifold, which can be viewed as a surface in \( \mathbb{R}^3 \). Notation: \( M^2_{2g} \).

**Example 9.4.** \( M^2_0 = S^2 \). \( M^2_1 = T^2 \). \( M^2_2 \) is a ‘pretzel’. Surfaces of higher genus can be considered as “generalized pretzels”.

(There is also a classification statement for non-orientable surfaces: any such surface is obtained from the sphere by gluing in Möbius bands.)

By applying Theorem A.2, one can easily calculate the Euler characteristic of orientable surfaces.

**Example 9.5.** Show that for a sphere with g handles,

\[
\chi(M^2_g) = 2 - 2g.
\]

Indeed, one can immediately see that \( \chi(S^1) = 0 \) and \( \chi(S^1 \times [0, 1]) = 0 \). Therefore all reduces to the Euler characteristic of a sphere with holes. Since each hole can be viewed as a small triangle with the interior removed, making a hole decreases the number \( c_2 \), and thus \( \chi \), by 1. The claim immediately follows.

Therefore one can express the genus as

\[
g(M^2) = 1 - \frac{1}{2} \chi(M^2).
\]

Note that for compact orientable 2-manifolds, \( \chi(M^2) \) is always even and non-positive (except for \( S^2 \)). The Gauß–Bonnet formula can be re-written as

\[
\frac{1}{2\pi} \oint_{M^2_g} K \, dS = 2 - 2g,
\]

for surfaces of genus g.
Remark 9.2. By using the derivation formulae for surfaces $M^2 \subset \mathbb{R}^3$, one can relate the integrand of the Gauß–Bonnet formula with the Gaußian map $F: M^2 \to S^2$, which sends each point $x \in M$ to the outward unit normal vector $n(x) \in S^2$, where $S^2$ is considered as the unit sphere in $\mathbb{R}^3$. Namely,

$$F^*dS_{\text{sphere}} = K dS$$

where $dS_{\text{sphere}}$ stands for the area 2-form on the unit sphere. Then

$$\oint_{M^2} K dS = 4\pi \deg(F)$$

where $\deg(F) \in \mathbb{Z}$ is the degree of the Gaußian map. (The degree of a map between compact oriented manifolds of the same dimension is defined as the number of preimages of a generic point counted with signs of the respective Jacobians.) This gives an alternative approach to the Gauß–Bonnet theorem.

Example 9.6. Make a picture of a 2-torus $T^2 \subset \mathbb{R}^3$ and a similar picture of a ‘generalized pretzel’ $M_g^2 \subset \mathbb{R}^3$. Chose a unit vector $n \in \mathbb{R}^3$ and find the points of $M_g^2$ where $n$ is the outward normal vector. Count their number taking in account orientations and show that it is exactly $1 - g$. This will be the degree of the Gaußian map.

The Gauß–Bonnet theorem for 2-dimensional manifolds can be generalized to manifolds of any even dimension. Such a generalization is known as the Gauß–Bonnet–Chern theorem and is a particular case of the fundamental Atiyah–Singer index theorem.

## A Triangulation and Euler characteristic

In this appendix we recall the notion of a triangulation of a topological space and related concepts.

A simplex of dimension $k$ or a $k$-simplex is a body $\sigma \subset \mathbb{R}^k$ consisting of all points of the form

$$x = t_0x_0 + t_1x_1 + \ldots + t_kx_k$$

where

$$t_0 + t_1 + \ldots + t_k = 1 \quad \text{and} \quad t_i \geq 0 \text{ for all } i$$

for some given points $x_i \in \mathbb{R}^k$ in general position, meaning that they do not belong to a plane of dimension less than $k$. The points $x_i$ are called the vertices of $\sigma$. The
numbers \( t_i \) are known as \emph{barycentric coordinates} on a simplex. (Note that they are not independent.) Simplices of dimension \( k \) can be considered as subspaces of \( \mathbb{R}^N \) for any \( N \geq k \).

**Example A.1.** A 0-simplex is a point. A 1-simplex is a closed segment. A 2-simplex is a (solid) triangle. A 3-simplex is a tetrahedron.

A \emph{face} of a simplex is a subspace specified by setting some of the barycentric coordinates to zero. Any face is a simplex of smaller dimension. Vertices are particular case of faces, the 0-dimensional faces. Faces of dimension 1 are called \emph{edges}. It is convenient to count the simplex itself as a face. Then a \( k \)-simplex has faces of dimensions 0, 1, \ldots, \( k \). The number of \( i \)-faces of a \( k \)-simplex equals \( \binom{k+1}{i+1} \).

**Example A.2.** A triangle has three vertices, three edges and one 2-face.

A finite (geometric) \emph{simplicial complex} is a finite collection \( K \) of simplices in \( \mathbb{R}^N \) with the following two properties:

- It a simplex \( \sigma \) belongs to \( K \), so do all the faces of \( \sigma \);
- Two simplices \( \sigma, \tau \in K \) either do not intersect at all or intersect by a single common face.

The union of all simplices of a complex \( K \) is a compact subspace of \( \mathbb{R}^N \) called the \emph{body} of \( K \) and denoted \( |K| \).

**Definition A.1.** A finite \emph{triangulation} of a topological space \( X \) is a homeomorphism \( \varphi: |K| \to X \) for some finite simplicial complex \( K \). (\( X \) is automatically compact.)

A triangulation introduces into a topological space \( X \) a “combinatorial structure”, a decomposition into pieces homeomorphic to simplices. (It can be encoded by making a list of all vertices, which will be a finite set, with an indication of the subsets corresponding to simplices. Such a structure is sometimes called a \‘simplicial scheme’ or \‘abstract simplicial complex’.) This structure can be used for obtaining information about the topology of \( X \). Note that if \( X \) has a triangulation, it is by no means unique (e.g., one can subdivide simplices to get a “finer” triangulation.) Topological invariants defined in terms of a triangulation should be independent on a choice of triangulation. An example is given by the following notion.

**Definition A.2.** The \emph{Euler characteristic} of a simplicial complex \( K \) is the integer defined as

\[
\chi(K) = c_0(K) - c_1(K) + \ldots = \sum (-1)^k c_k(K)
\]

where \( c_k(K) \) denotes the number of \( k \)-simplices in \( K \). (The sum is finite because the number of all simplices is finite.)
In topology the following theorem is proved.

**Theorem A.1.** For simplicial complexes $K$ and $L$, if $|K| \cong |L|$, then

$$\chi(K) = \chi(L).$$

Therefore one can unambiguously speak of the Euler characteristic of a topological space admitting a triangulation, independently of a choice of triangulation.

**Example A.3.** Find the Euler characteristics for the following spaces: the segment $[0,1]$ (answer: 1); the circle $S^1$ (answer: 0); the sphere $S^2$ (answer: 2).

A useful tool for calculating Euler characteristic is the following “addition property”.

**Theorem A.2.** For simplicial complexes $K$ and $L$ in $\mathbb{R}^N$ such that $K \cup L$ is also a complex,

$$\chi(K \cup L) = \chi(K) + \chi(L) - \chi(K \cap L).$$

**Proof.** The corresponding equality holds for the numbers of simplices in each dimension:

$$c_k(K \cup L) = c_k(K) + c_k(L) - c_k(K \cap L).$$

Indeed, when we count simplices in $K \cup L$ by adding the numbers of them in $K$ and $L$, we count the simplices occurring in both $K$ and $L$, twice. Hence we have to subtract. The equality for the Euler characteristic follows.

The condition that $K \cup L$ is a complex means that there are no ‘forbidden’ intersections of simplices from $K$ and $L$.

This theorem can be applied to topological spaces admitting triangulations.