§4 Tangent vectors and related objects. (Solutions.)

Problem 4.1 Consider the bases \( \{e_\theta\} \) and \( \{e_u\} \) for the tangent spaces to the circle \( S^1 \) corresponding to the polar angle \( \theta \) and the “stereographic coordinate” \( u \) respectively.

(a) Consider \( x = (x, y) \in S^1 \subset \mathbb{R}^2 \). For the angular coordinate \( \theta \), we have \( x = \cos \theta, y = \sin \theta \). Therefore

\[
e_\theta = \frac{dx}{d\theta} = (-\sin \theta, \cos \theta) = (-y, x).
\]

This is a vector in \( T_xS^1 \). It is a unit vector in the ambient Euclidean space \( \mathbb{R}^2 \) that can be obtained by rotating the radius-vector of \( x \) counterclockwise through 90°. For the stereographic coordinate \( u \) (where the north pole is chosen as the center) we have

\[
x = \frac{2u}{u^2 + 1}, \quad y = \frac{u^2 - 1}{u^2 + 1}.
\]

By differentiating we obtain

\[
e_u = \frac{dx}{du} = (\frac{-2u^2 + 2}{(u^2 + 1)^2}, \frac{4u}{(u^2 + 1)^2}) = \frac{2}{u^2 + 1} \left( \frac{1 - u^2}{u^2 + 1} \right) \left( 1 - y \right) \cdot (-y, x) = (1 - y) e_\theta.
\]

Hence the transformation law is: \( e_u = (1 - y) e_\theta \).

(b) Directly:

\[
e_\theta = \frac{du}{d\theta} e_u = \frac{d}{d\theta} \left( \frac{\cos \theta}{1 - \sin \theta} \right) e_u = \frac{-(1 - \sin \theta) \cos \theta - \cos \theta(- \cos \theta)}{(1 - \sin \theta)^2} e_u = \frac{1 - \sin \theta}{(1 - \sin \theta)^2} e_u = \frac{1}{1 - \sin \theta} e_u,
\]

or \( e_u = (1 - \sin \theta) e_\theta = (1 - y) e_\theta \) as obtained above. Here \( x = (x, y) = (\cos \theta, \sin \theta) \in S^1 \).

(c) The vector \( e_\theta \in T_xS^1 \) is well-defined and not vanishes at all points \( x \in S^1 \); hence it can be taken as a basis vector of the tangent space \( T_xS^1 \) for all \( x \in S^1 \). Though the vector \( e_u \in T_xS^1 \) makes sense for all points \( x \in S^1 \) as well, it vanishes at \( y = 1 \), i.e., at the point \( N = (0, 1) \in S^1 \); hence it can be taken as a basis of \( T_xS^1 \) for all \( x \in S^1 \setminus \{N\} \).

Problem 4.2 Directly: \( e_x = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} (x, y, z) = \frac{\partial}{\partial y} (x, y, f(x, y)) = (1, 0, f_x) \) and similarly \( e_y = \frac{\partial}{\partial y} = \frac{\partial}{\partial y} (x, y, z) = \frac{\partial}{\partial y} (x, y, f(x, y)) = (0, 1, f_y) \). (Here \( f_x \) and \( f_y \) denote the partial derivatives of \( f \) w.r.t. \( x \) and \( y \).)

Problem 4.3 From the formulas for \( x = (x, y, z) \in S^2 \subset \mathbb{R}^3 \),

\[
x = \cos \varphi \sin \theta, \quad y = \sin \varphi \sin \theta, \quad z = \cos \theta,
\]

we obtain the vectors \( e_\theta, e_\varphi \),

\[
e_\theta = \frac{dx}{d\theta} = (\cos \varphi \cos \theta, \sin \varphi \cos \theta, - \sin \theta),
\]

\[
e_\varphi = \frac{dx}{d\varphi} = (- \sin \varphi \sin \theta, \cos \varphi \sin \theta, 0)
\]

as elements of \( \mathbb{R}^3 \). We may note that at each point \( x \in S^2 \) different from the north pole \( N = (0, 0, 1) \), \( e_\theta \) is tangent to the meridian passing through this point (a meridian is a curve
Therefore \( \sin \theta \).

**Problem 4.8**

\( v \in \mathbb{R} \) or \( x = S = (0,0,-1) \). The vector \( e_\theta \) is not defined. (At \( N \) or \( S \), a meridian is not defined; there are many meridians starting from each of the poles.) At the same time, the vector \( e_\varphi \) is tangent to the parallel (the curve with \( \theta = \text{const} \)) through \( x \). Note that \( e_\varphi \) is zero at \( x = N \) or \( x = S \).

To summarize, the vectors \( e_\theta, e_\varphi \) are defined and make a basis of \( T_x S^2 \) for all points \( x \) of the sphere except for \( N \) and \( S \).

Consider now the point \( P = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \in \mathbb{R}^3 \). It belongs to \( S^2 \), so it makes sense to consider \( T_P S^2 \). Because the vector \( v = (0,1,-1) \in \mathbb{R}^3 \) is orthogonal to the radius-vector \( OP \in \mathbb{R}^3 \), we see that \( v \in T_P S^2 \) and can be expanded over the basis \( e_\theta, e_\varphi \). Practically we need to specify first the expressions for \( e_\theta, e_\varphi \) for the point \( P = (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \). We have \( \cos \theta = \frac{1}{\sqrt{3}} \), therefore \( \sin \theta = \frac{\sqrt{2}}{\sqrt{3}} \). Hence \( \cos \varphi = -\frac{1}{\sqrt{2}} \), \( \sin \varphi = \frac{1}{\sqrt{2}} \). This gives explicit expressions

\[
e_\theta = (\cos \varphi \cos \theta, \sin \varphi \cos \theta, -\sin \theta) = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{\sqrt{2}}{\sqrt{3}} \right),
\]

\[
e_\varphi = (-\sin \varphi \sin \theta, \cos \varphi \sin \theta, 0) = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0 \right)
\]
at \( P \in S^2 \). To obtain the coefficients of the expansion of \( v \) we write \( v = ae_\theta + be_\varphi \) where \( a, b \in \mathbb{R} \), and arrive at a system of three simultaneous equations for the two unknowns \( a \) and \( b \):

\[
\begin{cases}
-\frac{1}{\sqrt{6}} a - \frac{1}{\sqrt{3}} b = 0 \\
\frac{1}{\sqrt{6}} a - \frac{1}{\sqrt{3}} b = 1 \\
\frac{\sqrt{2}}{\sqrt{3}} a + 0 b = 0
\end{cases}
\]

This system is consistent precisely because \( v \in T_P S^2 \). By solving it, we find the (unique) numerical coefficients \( a = \frac{\sqrt{3}}{\sqrt{2}} \) and \( b = -\frac{\sqrt{3}}{2} \). The numbers \( \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2} \) are the components of the tangent vector \( v \) at the point \( P \in S^2 \) relative to the coordinate system \( \theta, \varphi \) on \( S^2 \).

**Problem 4.7**

The tangent map \( dF \) is defined by its matrix in some bases. Relative to the bases associated with standard coordinates on \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \), which we denote \( u, v \) and \( x, y, z \) respectively, the matrix is

\[
\frac{\partial x/\partial u}{\partial y/\partial u} \quad \frac{\partial x/\partial v}{\partial y/\partial v} \quad \frac{\partial z/\partial u}{\partial z/\partial v} = \begin{pmatrix} \cos v & -u \sin v & \\
\sin v & u \cos v & \\
0 & 1 & 
\end{pmatrix}.
\]

The images of the basis vectors \( e_1 = (1,0) \) and \( e_2 = (0,1) \) are precisely the columns of this matrix. Hence \( dF(u,v)(e_1) = \cos v e_1 + \sin v e_2 \) and \( dF(u,v)(e_2) = -u \sin v e_1 + u \cos v e_2 + e_3 \).

**Problem 4.8**

If we consider the real part \( x \) and the imaginary part \( y \) of a point \( z = x + iy \in \mathbb{C} \) as coordinates on \( \mathbb{C} \), then the matrix of the differential \( df(z) \) can be found as follows. We have \( \frac{\partial z^n}{\partial x} = nz^{n-1} \frac{\partial z}{\partial x} = nz^{n-1} \) and \( \frac{\partial z^n}{\partial y} = nz^{n-1} \frac{\partial z}{\partial y} = i nz^{n-1} \). Using the polar form of a complex number, we get \( nz^{n-1} = nr^{n-1}(\cos(n-1)\theta + i \sin(n-1)\theta) \). Hence

\[
\frac{\partial z^n}{\partial x} = nr^{n-1}(\cos(n-1)\theta + i \sin(n-1)\theta)
\]

\[
\frac{\partial z^n}{\partial y} = nr^{n-1}(-\sin(n-1)\theta + i \cos(n-1)\theta)
\]
so, separating the real and imaginary parts, we have, for the matrix of \( df \):

\[
\begin{pmatrix}
\frac{\partial \Re(z^n)}{\partial x} & \frac{\partial \Re(z^n)}{\partial y} \\
\frac{\partial \Im(z^n)}{\partial x} & \frac{\partial \Im(z^n)}{\partial y}
\end{pmatrix} = \begin{pmatrix}
nr^{n-1} \cos(n-1)\theta & nr^{n-1} \sin(n-1)\theta \\
-nr^{n-1} \sin(n-1)\theta & nr^{n-1} \cos(n-1)\theta
\end{pmatrix} - nr^{n-1} \begin{pmatrix}
\cos(n-1)\theta & \sin(n-1)\theta \\
-\sin(n-1)\theta & \cos(n-1)\theta
\end{pmatrix}.
\]

We see that if we identify tangent vectors at different points of \( \mathbb{C} \) with elements of the same space \( \mathbb{R}^2 \), the linear transformation \( df(z) \) where \( z = re^{i\theta} \) is the combination of the rotation through the angle \((n-1)\theta\) and dilatation with the coefficient \( nr^{n-1} \). (This is the same as the multiplication of vectors of \( \mathbb{R}^2 \cong \mathbb{C} \) by the complex number \( nz^{n-1} \). The fact that the differential of the map \( z \mapsto z^n \) has the form of the multiplication operator by a complex number is an expression of the holomophicity or complex-differentiability of this map.)

**Problem 4.4** We have \( AA^T = E \) as the equation specifying \( O(n) \), hence the equation specifying \( T_AO(n) \) is \( AA^T + AA^T = 0 \). It is equivalent to \( AA^T + (AA^T)^T = 0 \), i.e., to the condition that \( X = AA^T = AA^{-1} \) is antisymmetric. In particular, for \( T_EO(n) \) we have the equation \( \dot{A}^T = -\dot{A} \).

**Problem 4.5** (a) Consider \( ad - bc = 1 \) and differentiate. We arrive at \( \dot{a}d + a\dot{d} - bc - \dot{b}c = 0 \) as the equation of \( T_A\text{SL}(2) \). In particular, for \( A = E \) we obtain \( \dot{a} + \dot{d} = 0 \), which is \( \text{tr} X = 0 \) (if \( \dot{A} = X \)).

(b) Consider \( \det(E + \Delta t X) \) and expand it in \( \Delta t \). It is clear that the input into the term linear in \( \Delta t \) is given by the diagonal elements only, and we arrive at \( \det(E + \Delta t X) = 1 + \Delta t \text{ tr} X + \ldots. \) Hence the equation for \( T_E\text{SL}(n) \) is \( \text{tr} X = 0 \) (the same as for \( n = 2 \)).

**Problem 4.6** Show that the tangent bundle \( TS^1 \) for the circle is diffeomorphic to \( S^1 \times \mathbb{R} \).

(\textbf{Remark.} For a 2-sphere \( S^2 \), the analog of the above is not true: the tangent bundle \( TS^2 \) is \textit{not} the product of \( S^2 \) and \( \mathbb{R}^2 \).)

Consider the following map \( S^1 \times \mathbb{R} \to TS^1 : \)

\[
(x, t) \mapsto (x, t e_\theta(x))
\]

where \( x \in S^1 \) and \( e_\theta = \frac{\partial x}{\partial \theta} \) (see Problem \textit{??}, part (b)). The inverse map sends \( (x, v) \in TS^1 \) to \( (x, t) \in S^1 \times \mathbb{R} \), where \( t \in \mathbb{R} \) is obtained from \( v = t \cdot e_\theta(x) \).

**Problem 4.9** We know that the tangent space at each \( A \in O(n) \) consists of the matrices \( X \) of the form \( XA^T = B \) where \( B^T = -B \) (we may view \( B \) as a vector in \( T_EO(n) \)). Therefore the map \( TO(n) \to O(n) \times T_EO(n) \) that sends \( (A, X) \in TO(n) \) to \( (A, B = XA^T) \) is the desired diffeomorphism. Note also that for orthogonal matrices \( A^T = A^{-1} \), so we may re-write this map as \( (A, X) \mapsto (A, B) \) where \( B = XA^{-1} \).

**Problem 4.10** (a) We have \( (e^X)^T = e^{X^T} = e^{-X} = (e^X)^{-1} \), as claimed.

(b) We have \( \det e^X = e^{\text{tr}X} = e^0 = E \), as claimed. (We have used the Liouville formula \( \det e^A = e^{\text{tr} A} \).)
Consider a curve $X(t)$ passing through the zero matrix at $t = 0$ and with the velocity $A$ (some fixed matrix), for example, $X(t) = tA$. Then the differential $d \exp(0)$ maps $A$ to the velocity of the curve $e^{X(t)}$ at $t = 0$. Taking, for example, $X(t) = tA$ (the result, of course, does not depend on a choice of a curve), we obtain

$$d \exp(0)(A) = \left. \frac{d}{dt} e^{tA} \right|_{t=0} = \left. \frac{d}{dt} \right|_{t=0} \left( E + tA + \frac{1}{2} t^2 A^2 + \ldots \right) = A,$$

as claimed.

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