§2 Recollection of elementary functions: transcendental functions

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In this section we continue recollection of elementary functions. In particular, we consider exponential, trigonometric and hyperbolic functions, and their inverses (logarithmic function, “arc” and “area” functions: arcsin \( x \), arctan \( x \), arsinh \( x \), etc.). Such functions are together known as “transcendental”, which means they are not described algebraically, i.e. in terms of polynomials. (Note that power function discussed above when the power is real should properly be listed among transcendental functions, as it is expressed via the exponential and logarithmic functions.)

2.1 Exponentials and logarithms

When we define a power function, we consider an expression \( a^b \) for a fixed \( b \) and allow \( a \) to vary, treating it as the argument. If in the same expression \( a^b \) (where \( a > 0 \) is a positive real number) we fix \( a \) and allow \( b \) to vary (\( b \) can be any real number), we obtain a different type of functional dependance called “exponential”. Using \( x \) for the independent variable and \( y \) for dependent variable, we arrive at an exponential function

\[ y = a^x. \]

The fixed number \( a \) is called the base. Here we shall assume that \( a \) is not only positive, but also \( a \neq 1 \). (Since \( 1^x = 1 \) for all \( x \), so an ‘exponential’ with base 1 would be just constant. We exclude this case.)

Example 2.1. Sketch the graph of the function \( y = a^x \). There are two cases: \( a > 1 \) when \( y = a^x \) increases (think of \( y = 2^x \) for concreteness), and \( 0 < a < 1 \) when \( y = a^x \) decreases (think of \( y = \left(\frac{1}{2}\right)^x = \frac{1}{2^x} \) for concreteness).

Graph for various values of \( a \) all have the same basic shape. All the graphs pass through the point \((0, 1)\), but some rise or fall more steeply than others. Consider the behavior near the point \((0, 1)\). For \( a \) very large, the graph is very steep and with the growth of \( a \) the tangent at \((0, 1)\) becomes more and more vertical (slope almost \(+\infty\)); at the same time, for \( a \) just slightly larger than 1, the graph is almost flat at this point (slope almost 0). Therefore there must be some intermediate position of the tangent (for some particular value of \( a \)), where the slope is 1.

This value of \( a \) is called \( e \).

One may estimate the value of this constant. By trying \( a = 2 \) and \( a = 3 \) (and calculating approximately the slope of the graphs at \( x = 0 \)), one can see that \( e \) is between 2 and 3. One can develop methods for calculating \( e \) to any prescribed degree of precision. With eight decimal places, \( e = 2.71828182 \ldots \).

A strong statement is that \( e \) is not a rational number. Even more, it is (like \( \pi \)) not even an algebraic number, that is, it is not a root of a polynomial with rational coefficients\(^1\).

\[^1\text{Unlike } \sqrt{2}, \text{ for example. Among real numbers one distinguishes rational numbers and irrational numbers.}\]
As we shall later see, for any exponential function \( y = a^x \), the slope of the graph at each point \( x \) is proportional to the value of \( y \) (the larger is \( y \), the faster the function \( y = a^x \) grows or decays), with some constant coefficient of proportionality. The case \( a = e \) is therefore special because for \( y = e^x \) the slope at each \( x \) is exactly \( y \) itself (so that at \( x = 0 \), the slope is 1, which the definition of the number \( e \)).

The number \( e \) crops up all over the place in mathematics, as you’ll see even just on this course.

We can use functions of the form \( y = Ce^{kx} \) where \( C, k \) are constants\(^2\) to model exponential growth or exponential decay. (Question: for which combinations of \( k \) and \( C \) it is growth and for which, decay?) This growth and decay is very rapid and outruns any power function.

For example we can use a calculator to find (approximately) the values of \( y = x^{10} \cdot e^{-x} \) for different values of \( x \). (Incidentally, for writing very big and very small numbers, we use the so-called “scientific notation” where instead of e.g. 9876 we write: \( 9.876 \times 10^3 \). In general, it is \( m \times 10^n \) where \( 1 \leq |m| < 10 \). In the following, we keep just two digits after the decimal point.)

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
\hline
x & 0 & 5 & 10 & 15 & 20 & 25 & 30 & 35 & 40 \\
\hline
x^{10} & 0 & 9.77 \times 10^6 & 1 \times 10^{10} & 5.77 \times 10^{11} & 1.02 \times 10^{13} & 9.54 \times 10^{13} & 5.90 \times 10^{14} & 2.76 \times 10^{15} & 1.05 \times 10^{16} \\
\hline
e^{-x} & 1 & 6.74 \times 10^{-3} & 4.54 \times 10^{-5} & 3.06 \times 10^{-7} & 2.06 \times 10^{-9} & 1.39 \times 10^{-11} & 9.36 \times 10^{-14} & 6.30 \times 10^{-16} & 4.25 \times 10^{-18} \\
\hline
x^{10}e^{-x} & 0 & 6.59 \times 10^4 & 4.54 \times 10^5 & 1.77 \times 10^5 & 2.10 \times 10^4 & 1.33 \times 10^3 & 55.224 & 1.74 \times 10^2 & 4.46 \times 10^{-2} \\
\hline
\end{array}
\]

Initially the power function \( x^{10} \) keeps the value high but, as \( x \) increases, the exponential decay factor \( e^{-x} \) wins and the product rapidly approaches 0.

The notation \( \exp(\text{something}) \) is sometimes used when the “something” is better displayed in-line than as a superscript.

**Example 2.2.** 1. Find (using calculator) approximate values of the following to 3 decimal places.

\( (i) \ 2^{-8} \quad (ii) \ (5.1)^4 \quad (iii) \ (0.2)^{-3} \quad (iv) \ 3^{1/2} \quad (v) \ e^{-\pi}. \)

2. Plot \( y = x^3 \) and \( y = e^x \) on the same graph for \( 0 \leq x \leq 5 \). For which values of \( x \) is \( e^x > x^3 \)?

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\(^2\text{equivalently } y = Ca^x \text{ where } C, a \text{ are constants.}\)

---

Those irrational numbers that are roots of polynomials with rational coefficients are “not far” from rationals; they are known as **algebraic numbers**. Now, the irrational numbers that are **not** algebraic are called **transcendental**. In a certain precise sense, such are the “majority” of irrational numbers. However, for a given number — such as \( \pi \) or \( e \) — to prove that it is transcendental, is typically a very difficult problem.
(Note that we need to use different scales for the x-axis and y-axis. Otherwise it will be impossible to see anything from the graph. We observe that the graphs $y = e^x$ and $y = x^3$ intersect at two points, for $x = x_0$ between 1 and 2 and for $x = x_1$ between 4 and 5. Hence $e^x > x^3$ holds for $x < x_0$ and $x > x_1$.)

Finding logarithms is essentially the opposite of taking powers. Let $a$ and $b$ be positive real numbers. We define $\log_a b$, the logarithm to base $a$ of $b$, to be the power $x$ that $a$ has to be raised to to get $b$. We write that more mathematically:

$$\log_a b = x \text{ means that } a^x = b.$$ 

This can be rephrased as the following equations:

$$a^{\log_a b} = b \quad \log_a(a^x) = x.$$ 

The rules for combining logarithms are just consequences of the rules for combining powers.
• \( \log_a b + \log_a c = \log_a(bc) \)
• \( \log_a b - \log_a c = \log_a\left(\frac{b}{c}\right) \)
• \( \log_a(b^c) = c \log_a b \) in particular \( \log_a \frac{1}{b} = -\log_a b \)
• \( \log_a a = 1 \)
• \( \log_a 1 = 0 \)

Usually we work with logarithms to base 10 and base \( e \) (sometimes base 2 as well). Logs to base \( e \) are called natural logarithms and we write

\[ \ln b = \log_e b. \]

We can switch between bases using the change of base formula:

\[ \log_c b = \frac{\log_a b}{\log_a c} \]

If we look at the graph of a logarithmic function \( y = \ln x \) (or any \( y = \log_a x \) for \( a > 1 \)), we immediately see that it grows slower than linear function. (It is another expression of the fact that exponential grows faster than any power.)

Examples

1. Express the following as a single logarithm.

\[ \log_{10} 3 - 2 \log_{10} 5 + 0.5 \log_{10} 4 \]

2. Solve for \( x \).

\[ \log_2 x^2 - \log_2 (x + 1) = \log_2 (x + 2) \]

3. Solve the following equations, giving approximate answers to 3 decimal places. (You may need to use the change of base formula to calculate the answers.)

   (i) \( 2^x = 5 \)
   (ii) \( 5^{3y+1} = 8 \)
   (iii) \( e^{5t-1} = 0.01 \)
2.2 Trigonometric (or circular) functions

2.2.1 Measuring angles

Angles can be measured in degrees and in radians. Notation: \( a^\circ \) or \( b \text{ rad} \). The notation for the unit rad is commonly omitted. (So if no units are indicated, that means radians.) A full circle is defined to be \( 360^\circ \). In particular, it follows that a right angle (a quarter of a full circle) is \( 90^\circ \). Radians are less arbitrary units of angle because they are defined in terms of arc length. An angle of \( 1 \text{ radian} \) is defined to be the angle which makes an arc of length \( r \) on a circle of radius \( r \). Since the total arc length of a circle is \( 2\pi r \), there are \( 2\pi \) radians in a circle. So \( 2\pi \text{ rad} = 360^\circ \).

Angles are normally measured anti-clockwise from the \( x \)-axis as indicated.

This gives us a formula for converting between the two measurements.

\[
\text{angle in degrees} = \frac{180}{\pi} \times (\text{angle in radians})
\]

For example \( 90^\circ = \frac{\pi}{2} \text{ rads}, \) \( 30^\circ = \frac{\pi}{6} \text{ rads} \).

2.2.2 Definitions of trigonometric functions

Let \( ABC \) be a right angled triangle containing the angle \( \theta \) (exercise: draw and label a triangle to fit with the information below).

We define

\[
\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AC}{AB}
\]

\[
\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{BC}{AB}
\]

\[
\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} = \frac{AC}{BC}.
\]

Note that

\[
\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.
\]

The main trigonometric functions, which are sine and cosine, are defined as above for an acute angle \( \theta \), i.e., for \( 0 \leq \theta \leq \frac{\pi}{2} \). However, we may notice that \( \cos \theta \) and \( \sin \theta \) are respectively the \( x \)- and the \( y \)-coordinates of a point on the unit circle. That immediately allows to extend them to other values of the angles and in particular note their periodicity:

\[
\cos(\theta + 2\pi) = \cos \theta,
\]

\[
\sin(\theta + 2\pi) = \sin \theta,
\]

\[
\tan(\theta + 2\pi) = \tan \theta.
\]
Note some further useful relations:

\[
\begin{align*}
\sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta \\
\cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta \\
\sin\left(\frac{\pi}{2} + \theta\right) &= \cos \theta \\
\cos\left(\frac{\pi}{2} + \theta\right) &= -\sin \theta \\
\sin(\pi + \theta) &= -\sin \theta \\
\sin(\pi - \theta) &= \sin \theta \\
\cos(\pi \pm \theta) &= -\cos \theta
\end{align*}
\]

All of them can be seen from the diagram of a unit circle at the xy plane. Sine and cosine values lie between -1 and +1. Tangent values can take any value (and are undefined for certain values of \( \theta \)). Note that \( \cos \theta = \sin (\frac{\pi}{2} - \theta) \), \( \sin \theta = \cos (\frac{\pi}{2} - \theta) \). The angles \( \theta \) and \( \frac{\pi}{2} - \theta \) are called complementary. (Hence the names: cosine, i.e., ‘cosinus’ means ‘complementi sinus’.)

Useful special values:

\[
\begin{align*}
\sin(30^\circ) &= \cos(60^\circ) = \frac{1}{2}, & \sin(60^\circ) &= \cos(30^\circ) = \frac{\sqrt{3}}{2} \\
\sin(45^\circ) &= \cos(45^\circ) = \frac{1}{\sqrt{2}}, & \tan(45^\circ) &= 1
\end{align*}
\]

Example. Convert the following from radians to degrees: \( \frac{3\pi}{4} = 135^\circ \).

Examples. Find the exact value of the following:

(i) \( \sin(135^\circ) \) (ii) \( \cos(\frac{2\pi}{3}) \) (iii) \( \tan(-\frac{\pi}{3}) \)

Besides the main trigonometric functions, sine, cosine and tangent (in fact, it is sufficient to consider only sine and cosine), there are also

\[
\begin{align*}
\sec(\theta) &= \frac{1}{\cos(\theta)} \\
\cosec(\theta) &= \frac{1}{\sin(\theta)} \\
\cot(\theta) &= \frac{1}{\tan(\theta)}
\end{align*}
\]

Graphs of the trigonometric functions are as follows.
We can see that the trigonometric functions are periodic, meaning that the functions repeat the same values over a fixed period. Sine and cosine have period $2\pi$ and tangent has period $\pi$. 
Example 2.3.  1. Sketch the graphs of the functions $\sin(2x)$ and $\cos(x - \frac{\pi}{3})$. How do they differ from $\sin(x)$ and $\cos(x)$?

2. Sketch the graphs of cosec$(x)$, sec$(x)$ and cot$(x)$.

3. Find all the solutions of the following equations for $x$ in radians.

   (i) $\sin(x) = \frac{1}{2}$
   (ii) $\cos(x) = 0$
   (iii) $\tan(x) = -1$

2.2.3 Trigonometric identities

You should be familiar with the following result.

**Theorem 2.1** (Pythagoras Theorem). In a right-angled triangle, the sum of the square on the hypotenuse is equal to the sum of the squares on the other two sides, i.e.,

$$r^2 = x^2 + y^2$$

where $r$ is the length of the hypotenuse, $x$, $y$, the lengths of the other two sides.

We can use the definitions of the trigonometric functions, together with the Pythagoras Theorem to obtain the following main identities satisfied by trigonometric functions.

**Theorem 2.2.** For all values of the argument $\theta$,

$$\cos^2 \theta + \sin^2 \theta = 1.$$  

*Proof.* From the Pythagoras theorem we have

$$\cos^2 \theta + \sin^2 \theta = \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

**Corollary 1.** For all values of $\theta$ for which the functions are defined:

$$1 + \tan^2 \theta = \sec^2 \theta$$
$$\cot^2 \theta + 1 = \cosec^2 \theta$$

*Proof.* These formulas are obtained by dividing throughout by $\cos^2 \theta$ and $\sin^2 \theta$ respectively.

Further identities are based on the following “addition formulas” (or “compound angle” formulas).

**Theorem 2.3** (Addition formulas). For any angles $A$ and $B$,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
**Theorem 2.4** (Addition formulas for tangent).

\[
\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}
\]

\[
\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}
\]

**Proof.** Use the addition formula for sine and cosine:

\[
\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B - \sin A \sin B} = \frac{\frac{\sin A \cos B + \sin B \cos A}{\cos A \cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}
\]

The second formula follows in the same way. \qed

**Theorem 2.5** (Double angle formulas).

\[
\sin 2\theta = 2\sin \theta \cos \theta
\]

\[
\cos 2\theta = \cos^2 \theta - \sin^2 \theta
\]

The latter identity may also be written

\[
\cos 2\theta = 2\cos^2 \theta - 1 = 1 - 2\sin^2 \theta.
\]

**Proof.** Let \( \theta = A = B \) in the sum identities

\[
\sin 2\theta = \sin (\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta = 2\sin \theta \cos \theta
\]

\[
\cos 2\theta = \cos (\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta = \cos^2 \theta - \sin^2 \theta
\]

Using the identity \( \cos^2 \theta + \sin^2 \theta = 1 \) to eliminate either \( \cos^2 \theta \) or \( \sin^2 \theta \) from the identity for \( \cos 2\theta \) completes the proof. \qed

**Corollary 2.** \( \tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)} \)

It is possible to deduce general formulas for \( \cos nx \) and \( \sin nx \). We shall not do that, but consider particular examples instead.

**Example 2.4.** \( \cos 3x = \cos(2x + x) = \cos 2x \cos x - \sin 2x \sin x = (\cos^2 x - \sin^2 x) \cos x - 2 \sin x \cos x \sin x = (2 \cos^2 x - 1) \cos x - 2 \cos x (1 - \cos^2 x) = 4 \cos^3 x - 3 \cos x \). Therefore

\[
\cos 3x = 4 \cos^3 x - 3 \cos x.
\]

**Example 2.5.** \( \cos 4x = \cos(3x + x) = (4 \cos^3 x - 3 \cos x) \cos x - (3 \sin x - 4 \sin^3 x) \sin x = 4 \cos^4 x - 3 \cos^2 x - 3 \sin^2 x + 4 \sin^4 x = 4 \cos^4 x + 4 \sin^4 x - 3 = 4 \cos^4 x + 4(1 - \cos^2 x)^2 - 3 = 4 \cos^4 x + 4(1 - 2 \cos^2 x + \cos^4 x) - 3 = 8 \cos^4 x - 8 \cos^2 x + 1 \). Therefore

\[
\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1.
\]

**Example 2.6.** By using the addition formula show that

\[
\sin(75^\circ) = \frac{1 + \sqrt{3}}{2\sqrt{2}}.
\]

**Solution.** We have \( 75^\circ = 30^\circ + 45^\circ = \frac{\pi}{6} + \frac{\pi}{4} \). Hence \( \sin(75^\circ) = \sin(\frac{\pi}{6} + \frac{\pi}{4}) = \sin \frac{\pi}{6} \cos \frac{\pi}{4} + \cos \frac{\pi}{6} \sin \frac{\pi}{4} = \frac{1}{2} \frac{\sqrt{3}}{2} + \frac{1}{2} \frac{\sqrt{3}}{2} = \frac{1 + \sqrt{3}}{2\sqrt{2}}. \)
2.2.4 Solving trigonometric equations

From the graphs of \( \sin(x) \), \( \cos(x) \) and \( \tan(x) \) we can see that there are infinitely many values of \( x \) that give the same value of the function. This causes problems when we want to go back from the value of the function to the value of the angle that it came from - because there are infinitely many angles that could have given that value of the function. For example how do we solve

\[
\sin(x) = 0?
\]

There are infinitely many solutions to this equation, \( x = 0, \pm \pi, \pm 2\pi, \ldots \)

We can define an inverse operation to taking sine if we choose a standard interval of width \( \pi \) and restrict the values that this inverse can take to that interval. We’ll choose our standard interval for the inverse sine function

\( \text{arcsin} \) to run from \(-\frac{\pi}{2}\) to \(\frac{\pi}{2}\).

That is, define the inverse sine or the \( \text{arc sine} \) function \( \text{arcsin}(x) \) to be the unique value \( y \) with \(-\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \) and \( \sin(y) = x \).

So \( \text{arcsin}(0.5) = \frac{\pi}{6} \) and \( \text{arcsin}(-0.8) \approx -0.93 \). (Note that \( \pi \approx 3 \), so \( \frac{\pi}{2} \approx 1.5 \).)

Similarly we can define inverse functions for cosine and tangent.

The inverse cosine or the \( \text{arc cosine} \) function, \( \text{arccos}(x) \), of \( x \) is the unique value \( y \) with \( 0 \leq y \leq \pi \) and \( \cos(y) = x \).

(Note we chose a different standard interval for this but, if you think about the shape of the graph of cos, it does make sense to do so.)

The inverse tangent or the \( \text{arc tangent} \) function, \( \text{arctan}(x) \), of \( x \) is the unique value \( y \) with \(-\frac{\pi}{2} < y < \frac{\pi}{2} \) and \( \tan(y) = x \).

Remark: notation such as \( \text{Arcos} \), \( \text{Arcsin} \), \( \text{Arctan} \) (with the capital A) is sometimes used for denoting ALL solutions of the corresponding trigonometric equation, as opposed to \( \text{arccos} \), etc. We have

\[
\text{Arcsin } x = (-1)^k \text{arcsin } x + k\pi, \ k = 0, \pm 1, \pm 2, \ldots
\]

Similarly,

\[
\text{Arccos } x = \pm \text{arccos } x + 2k\pi, \ k = 0, \pm 1, \pm 2, \ldots
\]

and

\[
\text{Arctan } x = \text{arctan } x + k\pi, \ k = 0, \pm 1, \pm 2, \ldots
\]

Note that \( \text{Arccos } x, \text{Arcsin } x, \text{Arctan } x \) are “multi-valued” and therefore cannot be regarded as genuine functions. (The names with “arc” originate from the fact that the angle coincides with the arc length for the unit circle used to define all trigonometric functions.)

When we solve equations involving trigonometric functions we should consider all possible solutions initially, not just those one our chosen interval. Some of these may be later discarded due to physical restrictions on the range of possible values (e.g. we may need only positive solutions).

For example consider the equation

\[
\cos(5y) = -0.5 \quad (\ast)
\]

\( \ast \)The notation \( \sin^{-1}(x) \) is also used in place of \( \text{arcsin}(x) \) but we avoid this, and corresponding notations for the other inverse trig functions, here because that notation might (wrongly!) suggest the meaning \( 1/\sin(x) \).
We have \( \arccos(-0.5) = \frac{2\pi}{3} \). From the unit circle (or from graph of \( \cos(x) \)) we see that we get the same value of \( \cos(x) = -0.5 \) for

\[
x = \frac{2\pi}{3} \pm 2n\pi \quad \text{and} \quad x = \frac{4\pi}{3} \pm 2n\pi \quad \text{for any} \quad n = 0, 1, 2, \ldots
\]

So the solution to (*) is

\[
y = \frac{2\pi}{15} \pm \frac{2n\pi}{5} \quad \text{and} \quad y = \frac{4\pi}{15} \pm \frac{2n\pi}{5} \quad \text{for any} \quad n = 0, 1, 2, \ldots
\]

This gives values \( y = \ldots, -\frac{4\pi}{15}, -\frac{2\pi}{15}, \frac{2\pi}{15}, \frac{4\pi}{15}, \frac{8\pi}{15}, \ldots \).

**Example 2.7.** Find all solutions to the following equations.

(i) \( \sin(4x) = \frac{1}{\sqrt{2}} \)

(ii) \( \tan(3x) = -1 \)

**Example 2.8.** Some frequently met values of the arc functions: \( \arcsin 0 = 0 \), \( \arccos 0 = \frac{\pi}{2} \), \( \arctan 0 = 0 \), \( \arcsin \frac{\sqrt{2}}{2} = \arccos \frac{\sqrt{2}}{2} = \frac{\pi}{4} \), \( \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3} \), \( \arcsin \frac{1}{2} = \frac{\pi}{6} \), \( \arctan 1 = \frac{\pi}{4} \).

**Example 2.9.** Let \( \cos \alpha = \frac{1}{2} \). Find all values of \( \alpha \).

**Solution:** We have \( \alpha = \arccos \frac{1}{2} = \pm \arccos \frac{1}{2} + 2k\pi = \pm \frac{\pi}{3} + 2k\pi \). Note that \( -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3} \), so the solution can be written in an alternative form \( \alpha = \frac{\pi}{3} + 2k\pi \) or \( \alpha = \frac{5\pi}{3} + 2k\pi \).

**Example 2.10.** Find \( \theta \) in the range \( 0 \leq \theta < \pi \) such that \( \tan \theta = 5 \).

**Solution:** The unique solution in the range between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \) is \( \theta = \arctan 5 \approx 1.37 \) (using a calculator or tables). General solution of the equation \( \tan \theta = 5 \) is obtained by adding integral multiples of \( \pi \). However, adding any multiple of \( \pi \) to \( \theta = \arctan 5 \) takes it out of the range \( 0 \leq \theta < \pi \). Hence the answer is: \( \theta = \arctan 5 \approx 1.37 \)

**Example 2.11.** Solve for all values: \( \cos x = \frac{1}{7} \).

**Solution:** We have \( \cos x = \frac{1}{\sqrt{7}} \) or \( \cos x = -\frac{1}{\sqrt{7}} \). Therefore \( x = \pm \arccos \frac{1}{\sqrt{7}} + 2k\pi \) or \( x = \pi \pm \arccos \frac{1}{\sqrt{7}} + 2k\pi \). This combines into \( x = \pm \arccos \frac{1}{\sqrt{7}} + k\pi \).

**Example 2.12.** Solve the equation: \( 2\sin^2 x + 5\sin x - 3 = 0 \).

**Solution:** We factorize: \( (2\sin x - 1)(\sin x + 3) = 0 \), so \( \sin x = \frac{1}{2} \) or \( \sin x = -3 \). The second equation has no solutions, so we have \( \sin x = \frac{1}{2} \iff x = (-1)^k \arcsin \frac{1}{2} + k\pi = (-1)^k \frac{\pi}{6} + k\pi \).

**Example 2.13.** Express \( 2\cos x + 3\sin x \) in the form \( A\sin(x + x_0) \) where \( A \) and \( x_0 \) are to be determined.

**Solution:** We can write

\[
2\cos x + 3\sin x = \sqrt{4 + 9} \left( \frac{2}{\sqrt{4 + 9}} \cos x + \frac{3}{\sqrt{4 + 9}} \sin x \right) = \sqrt{13} \left( \frac{2}{\sqrt{13}} \cos x + \frac{3}{\sqrt{13}} \sin x \right).
\]

Now we look for \( x_0 \) such that \( \cos x_0 = \frac{2}{\sqrt{13}} \) and \( \sin x_0 = \frac{3}{\sqrt{13}} \). We can take \( x_0 = \arcsin \frac{3}{\sqrt{13}} = \arccos \frac{2}{\sqrt{13}} \). So

\[
2\cos x + 3\sin x = \sqrt{13} \sin(x + x_0) \quad \text{where} \quad x_0 = \arcsin \frac{2}{\sqrt{13}}.
\]
2.3 Hyperbolic functions

Definitions:

\[
\cosh x = \frac{1}{2}(e^x + e^{-x}) \\
\sinh x = \frac{1}{2}(e^x - e^{-x})
\]

Then

\[
\tanh x = \frac{\sinh x}{\cosh x}.
\]

Main identity (check!):

\[
\cosh^2 x - \sinh^2 x = 1.
\]

Example 2.14. Sketch graphs of \(y = \sinh x\), \(y = \cosh x\) and \(y = \tanh x\). (Hint: sketch first \(y = \frac{1}{2}e^x\), \(y = \frac{1}{2}e^{-x}\), and \(y = -\frac{1}{2}e^{-x}\).) [See on a separate page.]

Note that \(\cosh(-x) = \cosh x\), \(\sinh x = -\sinh x\), \(\tanh(-x) = -\tanh x\). Also that \(\cosh x \geq 1\) and \(\sinh x\) takes all real values. As for \(\tanh x\), we have \(|\tanh x| < 1\) for all real \(x\).

A useful thing to note is that

\[
e^x = \cosh x + \sinh x, \\
e^{-x} = \cosh x - \sinh x.
\]

Example 2.15. Check the following formulas for \(\sinh(x + y)\), \(\cosh(x + y)\), \(\tanh(x + y)\) in terms of the same functions (addition formulas):

\[
\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y, \\
\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y.
\]

Also

\[
\tanh(x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}.
\]

Inverse hyperbolic functions: the (unique) solution \(y\) of the equation

\[
\sinh y = x
\]

(where \(x\) is arbitrary real number) is called the inverse hyperbolic sine or area sine, notation:

\[
y = \arsinh x
\]

(warning: unlike the circular case, it is pronounced not “arc sine”, but “area sine” because it is indeed related with a certain area).

Similarly is defined the area tangent, or inverse hyperbolic tangent \(\artanh x\): it is the solution \(y\) of the equation

\[
\tanh y = x.
\]

Inverse hyperbolic functions can be expressed in terms of logarithms (exercise!).
Example 2.16. To find \( y = \text{arsinh} x \) we write down \( \sinh y = x \) and solve for \( y \). We obtain:
\[
\frac{1}{2}(e^y - e^{-y}) = x \text{ or } e^y - e^{-y} = 2x \text{ or } e^{2y} - 1 = 2xe^y.
\]
Denoting \( e^y = z > 0 \), we obtain a quadratic equation for \( z \):
\[
z^2 - 2xz - 1 = 0.
\]
The discriminant \( D = (2x)^2 + 4 = 4(x^2 + 1) \) is positive, so we get the roots
\[
z_{1,2} = \frac{1}{2}(2x \pm 2\sqrt{x^2 + 1}).
\]
Note that \( z > 0 \), so we need to choose the “plus”. Hence \( z = x + \sqrt{x^2 + 1} > 0 \), and \( y \) is obtained by taking logarithm. Finally:
\[
\text{arsinh} x = \ln \left( x + \sqrt{x^2 + 1} \right)
\]
for all real \( x \).

Example 2.17. Solve the equation \( \cosh y = x \) for \( y \). First we note that \( x \) must be \( \geq 1 \) and that there are, in general, two solutions (positive and negative). By definition, \( \text{arcosh} x \) is the non-negative solution. One can show, similarly with the above, that
\[
\text{arcosh} x = \ln \left( x + \sqrt{x^2 - 1} \right)
\]
for all \( x \geq 1 \).

Example 2.18. Solve \( \tanh y = x \) for \( y \). First note that \( |x| < 1 \) for there to be a solution. Similarly to the above, we deduce
\[
\text{artanh} x = \frac{1}{2} \ln \frac{1 + x}{1 - x}
\]
for all \( x \) such that \( |x| < 1 \).
Graphs of hyperbolic functions:

A. \( y = \sinh x \) and \( y = \cosh x \)

First sketch \( y = \frac{1}{2} e^x \), \( y = \frac{1}{2} e^{-x} \), and \( y = -\frac{1}{2} e^{-x} \).

We see that \( y = \sinh x \) approaches \( y = \frac{1}{2} e^x \) for \( x \gg 1 \) (large positive), from below, and approaches \( y = -\frac{1}{2} e^{-x} \) for \( x \ll -1 \) (large negative), from above.

Also, \( y = \cosh x \) approaches \( y = \frac{1}{2} e^x \) for large positive \( x \), from above, and \( y = \frac{1}{2} e^{-x} \) for large negative \( x \), also from above.

B. \( y = \tanh x = \frac{e^{2x} - 1}{e^{2x} + 1} \)

We see that \( |y| < 1 \) always.

The graph \( y = \tanh x \) tends to \( y = 1 \) for large positive \( x \), and tends to \( y = -1 \) for large negative \( x \).