You met vectors in the first year. Vector calculus is essentially calculus on vectors. We will need to differentiate vectors and perform integrals involving vectors. In particular, we will look at two fundamental results called The Divergence Theorem and Stokes Theorem. Now would be an ideal time to revise any notes you have on vectors. Some of the basic facts that you should already know about vectors, and operations on vectors that involve derivatives (divergence $\nabla\cdot$, gradient ∇, curl $\nabla\times$) are summarised on the **handout on div, grad and curl**. Exercise Sheet 9 also contains questions that are intended as revision.

6.1. Introduction

A vector field $\mathbf{F}$ in three dimensions is a rule which tells us how to associate a vector with each point (x, y, z). See the **handout on div, grad and curl**. For example, the velocity of a fluid is a vector field. In general,

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k},$$

where F_x, F_y and F_z are functions of x, y, z. The handout shows the following two-dimensional examples (with $F_z = 0$).

$$\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}, \qquad \mathbf{F}_2 = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}.$$

The vector field

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

is called the radial direction vector. We will meet it several times. Note that $\mathbf{F}_1$ is the two-dimensional version.

Divergence and curl are two important mathematical operations on vector fields. Recall,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

The divergence gives a measure of 'net mass flow' If $\nabla \cdot \mathbf{F} = 0$ then no mass is created or destroyed. A simple calculation reveals that $\nabla \cdot \mathbf{F}_2 = 0$ (check!). If we focus on a small region in the x-y plane then we can see that mass is conserved by observing that the arrows pointing into that region are matched by arrows of the same length pointing out of that region.

The curl $\nabla \times \mathbf{F}$ gives a measure of twisting or 'curling' of a vector field. A simple calculation reveals that $\nabla \times \mathbf{F}_2 = 0$ (check!). The curl essentially tells us how a particle released into the flow field rotates. Recall, the curl is calculated via

$$\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i}, \frac{\partial}{\partial y}\mathbf{j}, \frac{\partial}{\partial z}\mathbf{k}\right) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}.
\end{aligned}$$

Example. Consider $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}$ then

$$\nabla \times \mathbf{F} = (xz - x)\mathbf{i} - yz\mathbf{j} + (z - x^2)\mathbf{k}.$$

There are two fundamental identities involving the operators div, grad and curl. For all scalar functions $f = f(x, y, z)$,

$$\nabla \times (\nabla f) = 0.$$

The curl of a gradient is always zero. In addition, for all vector fields $\mathbf{F}$,

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

The divergence of the curl of a vector is always zero. You are asked to prove these identities on Exercise Sheet 9.

6.2. Volume Integrals (of Scalar Functions)

Recall, the double integral

$$\int\int_D f(x, y)dA,$$

where D is a two-dimensional region in the x-y plane and $dA = dxdy$ (the area element in Cartesian coordinates) represents the volume between the surface $z = f(x, y)$ and the region D.

If D is a rectangle, then we have constant limits of integration. We can then easily swap the order of integration without worrying about the limits. For example, if $D = [0, 1] \times [0, 2]$ and $f(x, y) = x^2 + y^2$ then

$$\int\int_D f(x, y)\, dA = \int_{x=0}^{1}\int_{y=0}^{2} x^2 + y^2\, dydx = \int_{y=0}^{2}\int_{x=0}^{1} x^2 + y^2\, dxdy.$$

If D is a region with a more complicated shape, then we usually don't have constant limits of integration. Limits for the inner integral must be expressed as functions of the outer variables.

Example. Compute the integral

$$\int\int_D 1 - x - y\, dA,$$

where D is the right-angled triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. If we choose to perform the y integral first, then

$$\int\int_D 1 - x - y\, dA = \int_{x=0}^{1}\int_{y=0}^{1-x} 1 - x - y\, dydx.$$

Alternatively, if we perform the x integral first, then

$$\int\int_D 1 - x - y\, dA = \int_{y=0}^{1}\int_{x=0}^{1-y} 1 - x - y\, dxdy.$$

In both cases, you should find that the answer is 1/6. The important point is that we cannot just swap the order of integration and keep the same limits.

Now, a volume integral (of a scalar function f) over a three-dimensional volume V is denoted

$$\int\int\int_V f\, dV,$$

where dV is the so-called volume element. We use three integral signs here to emphasis that the integral is over a three-dimensional volume, but it is also ok to only write one integral sign. In Cartesian coordinates, you already know that the volume element is

$$dV = dxdydz.$$

Remember that an integral is really defined as a limit. We replace the integral with a sum over pieces of V. On each piece we evaluate the function f and multiply by the volume of the piece. We then take the limit of the sum as the number of pieces tends to infinity. In Cartesian coordinates, a natural way to break up a volume V is into small bricks. If the brick has lengths dx, dy and dz in each of the x, y and z coordinate directions then the volume of the piece is $dxdydz$. This is the 'volume element'.

If the volume V is a simple brick, then we have constant limits of integration. So, if $V = [a, b] \times [c, d] \times [e, f]$ then

$$\int\int\int_V f(x, y, z)\, dV = \int_{z=e}^{f} \int_{y=c}^{d} \int_{x=a}^{b} f(x, y, z)\, dxdydz.$$

In this case, we can swap the order of integration easily and we don't need to worry about changing the limits. As with double integrals, if V is more complicated then we need to pay attention to the limits of integration. Drawing a picture of V usually helps determine the correct limits!

Example. Evaluate

$$\int\int\int_V 1 + xy\, dV$$

where V is the tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 0)$ and $(1, 0, 0)$. Lets perform the z integral first, followed by the y integral and finally the x integral. If we perform the z integral first, then we need to provide limits for z as a function of the outer variables y and x. For a fixed x and a fixed y, the z coordinate is bounded by the face of the tetrahedron in the x-y plane (where $z = 0$) and the face of the tetrahedron that coincides with the plane $z = 1 - x - y$. Hence,

$$\int\int\int_V 1 + xy\, dV \quad = \quad \int_x \int_y \left(\int_{z=0}^{1-x-y} 1 + xy\, dz \right) dy\, dx.$$

The limits for y should then be expressed as functions of the outer variable x. We have

$$\int\int\int_V 1 + xy\, dV \quad = \quad \int_x \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1 + xy\, dz\, dy\, dx.$$

Finally, the variable x varies from 0 to 1 so

$$\int\int\int_V 1 + xy\, dV \quad = \quad \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 1 + xy\, dz\, dy\, dx = 7/40.$$

If V is not a brick but a more complicated shape that is not easy to describe in the Cartesian co-ordinate system, then it may be easier to work in an alternative coordinate system. If we do this, however, care must be taken to convert the volume element dV in the proper way. For instance, in cylindrical coordinates, dV does **not** mean $drd\theta dz$.

You already learned how to do double integrals in polar coordinates (revise your first year notes if you have forgotten). Recall,

$$\int\int_A f\, dA = \int_x \int_y f(x, y)\, dx\, dy = \int_r \int_\theta f(x(r, \theta), y(r, \theta))\, r\, dr\, d\theta.$$

The area element here in polar coordinates is $dA = rdrd\theta$. In Cartesian coordinates, we have $dA = dxdy$. Where does the extra factor of r come from? Recall that r is the determinant of the Jacobian matrix

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}.$$

It accounts for the change in area when we map a small rectangle with area $dxdy$ into polar co-ordinates. The mapped rectangle is not a rectangle.

In three dimensions, we know that the volume element for **Cartesian coordinates** is

$$dV = dxdydz.$$

In **cylindrical coordinates** we have

$$dV = rdrd\theta dz$$

and in **spherical coordinates** we have

$$dV = \rho^2 \sin\phi \, d\rho d\phi \, d\theta.$$

You should learn these formulae. To work out the last two explicitly, you can write down the 3×3 Jacobian matrix and find its determinant, exactly as you did for polar co-ordinates.

Let's test out the claim that $dV = \rho^2 \sin\phi \, d\rho d\phi \, d\theta$ for spherical co-ordinates. Note that

$$\int\int\int_V 1dV$$

gives the volume of V. If we get the volume element dV and the limits of integration right, then integrating one over a sphere should give the volume of the sphere (which you already know how to compute).

Example. Find the volume of a sphere of radius two. Integrating one over V where V is the sphere centred at the origin with radius two gives

$$\begin{aligned}
\int\int\int_V 1\,dV &= \int\int\int_V 1\,\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
&= \int_{\phi=0}^{\pi} \int_{\theta=0}^{2\pi} \left(\frac{8\sin\phi}{3} \right) d\theta \, d\phi \\
&= \int_{\phi=0}^{\pi} \frac{16\pi \sin\phi}{3} \, d\phi \\
&= -\frac{16\pi\cos(\pi)}{3} + \frac{16\pi\cos(0)}{3} \\
&= \frac{16\pi}{3} + \frac{16\pi}{3} = \frac{32\pi}{3}.
\end{aligned}$$

Now, of course, we also know that the standard formula for the volume of a sphere is $\frac{4}{3}\pi\rho^3$ where ρ is the radius. Since the radius is two, the volume is $\frac{4}{3} \times \pi \times 8 = \frac{32}{3}\pi$, which matches the above calculation.

There is an important theorem that connects: the divergence $\nabla \cdot \mathbf{F}$ of a vector field, a closed volume V, and its surface S. The theorem is quite technical. We present it first, and then investigate it.

Theorem: The Divergence Theorem. Let V be a bounded, closed region in space with piecewise smooth boundary S. Let $\hat{\mathbf{n}}$ be the unit normal vector to S, pointing **outward**. Then, if $\mathbf{F}$ is a differentiable vector field,

$$\int\int\int_V \nabla\cdot\mathbf{F}\,dV = \int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS.$$

In essence, this result says that the total divergence of a vector field in a bounded region V in space is equal to the net flow (or 'flux') across the boundary of the surface in the normal direction. Note that $\nabla\cdot\mathbf{F}$ is a scalar function so the integral on the left-hand side is a standard volume integral (which you know how to evaluate). Note that $\mathbf{F}\cdot\hat{\mathbf{n}}$ is the dot product of two vectors and this is also a scalar function. So, to evaluate the right-hand side of the equation, we first need to know how to find the normal vector to the surface S, and then how to evaluate surface integrals of scalar functions.

Unit Normal Vectors to Surfaces

For some surfaces, it is easy to determine the unit normal vector.

Example. Consider the unit cube. That is, the cube whose edges all have length one, and one of the vertices is the origin $(0,0,0)$. The cube has six faces, which are portions of the surfaces $z=0$, $z=1$ (the top and bottom faces), $x=0$, $x=1$, $y=0$ and $y=1$ (the side faces). The cube is a close volume. The unit vector that points out of the cube at each of the six faces is aligned with one of the x, y and z coordinate axes. On the top surface, the vector must point straight up (and have length one), so $\hat{\mathbf{n}}=\mathbf{k}$. On the bottom surface, the vector must point straight down (and have length one), so $\hat{\mathbf{n}}=-\mathbf{k}$. Similarly, on the four side faces, we have $\hat{\mathbf{n}}=-\mathbf{i}$, $\hat{\mathbf{n}}=\mathbf{i}$, $\hat{\mathbf{n}}=-\mathbf{j}$, and $\hat{\mathbf{n}}=\mathbf{j}$.

For curved surfaces it is more complicated to determine $\hat{\mathbf{n}}$. For surfaces of the form $z=f(x,y)$, we have the following general formula.

Let S be a portion of a surface of the form $z=f(x,y)$. A unit normal vector to S is

$$\hat{\mathbf{n}} = \frac{-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2}}.$$

Note that this vector is simply 'a' normal vector to the surface. If we reverse the sign, then it is still pointing in a direction normal to the surface. For closed surfaces, if we want the outward pointing normal, we must pay attention to the sign.

Example. Let S be the surface of the unit sphere. On S, we have $x^2+y^2+z^2=1$. So, the surface can be expressed as

$$z = \pm\sqrt{1-x^2-y^2} = f(x,y).$$

The positive square root corresponds to points on the upper hemisphere and the negative square root corresponds to the lower hemisphere. Differentiating gives

$$\frac{\partial f}{\partial x} = -\frac{x}{z}, \quad \frac{\partial f}{\partial y} = -\frac{y}{z},$$

(for both signs) and so

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \frac{1}{z}\sqrt{z^2+x^2+y^2}.$$

Using the above formula, we have

$$\hat{\mathbf{n}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2 + x^2 + y^2}}.$$

We know that on S, $x^2 + y^2 + z^2 = 1$. Hence, a unit normal vector to S is

$$\hat{\mathbf{n}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

This is, of course, the radial position vector. It points in the outward normal direction at all points on the surface of the sphere.

6.3. Surface Integrals

Let S be a portion of a surface $z = f(x, y)$. Note that here S could be open or closed. The integral

$$\int\int_S 1\, dS$$

is the surface area of S and

$$\int\int_S G(x, y, z)\, dS$$

is the surface integral of G. If the surface is of the form $z = f(x, y)$ then we can convert the surface integral into a standard double integral over a flat region D in the x-y plane by performing a change of variable. To do this, we first need to relate a small area δS on the (possibly curved) surface S to its projection δD (or 'shadow') onto the x-y plane.

It can be shown that the relationship between δS and δD is

$$\delta S = \frac{\delta D}{\hat{\mathbf{n}} \cdot \mathbf{k}}$$

where $\hat{\mathbf{n}}$ is the unit normal vector to S on δS (and obviously $\mathbf{k}$ is the unit normal vector to δD, pointing up). So,

$$\int\int_S G(x, y, z)\, dS = \int\int_S G(x, y, z(x, y)) \frac{1}{\hat{\mathbf{n}} \cdot \mathbf{k}}\, dxdy.$$

Using the formula for $\hat{\mathbf{n}}$, we have

$$\hat{\mathbf{n}} \cdot \mathbf{k} = \frac{1}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

and so,

$$\int\int_S G(x, y, z)\, dS = \int\int_S G(x, y, z(x, y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy.$$

Basically, when converting to a standard integral in the x-y plane, we just need to remember the square root factor to account for the change in curvature. Note that if S is itself flat, and parallel to the x-y plane, then $\hat{\mathbf{n}} = \mathbf{k}$ and $\hat{\mathbf{n}} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{k} = 1$, so there is no extra factor.

Example. Evaluate the surface integral $\int\int_S z^2\, dS$, where S is the open surface corresponding the portion of the surface of the unit sphere that lies in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$). **[Note: it might help to draw a diagram of S here.]**

S is a portion of the surface

$$z = +\sqrt{1-x^2-y^2} = f(x,y).$$

We have already seen that

$$\frac{\partial f}{\partial x} = -\frac{x}{z}, \quad \frac{\partial f}{\partial y} = -\frac{y}{z}$$

and so

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \frac{1}{z}\sqrt{z^2+x^2+y^2} = \frac{1}{z}$$

on S. Hence,

$$\int\int_S z^2\,dS = \int\int_D z^2\frac{1}{z}\,dx\,dy = \int\int_D z\,dx\,dy = \int\int_D \sqrt{1-x^2-y^2}\,dx\,dy.$$

This is now just a standard double integral but we need to work out the limits of integration. D is the projection of S onto the x-y plane. This is a quarter circle bounded by the lines $x=0$, $y=0$ and $x^2+y^2=1$. It would be easier to evaluate this integral in polar co-ordinates. So, changing variables once again,

$$\int\int_S z^2\,dS = \int_{\theta=0}^{\pi/2}\int_{r=0}^{1}\sqrt{1-r^2}\,r dr d\theta.$$

(Don't forget the extra factor of r in the area element when we change variables). The integral with respect to r can be done by substitution or by just identifying a function whose derivative is the given integrand

$$\int\int_S z^2\,dS = \int_{\theta=0}^{\pi/2}\left[-\frac{\left(1-r^2\right)^{3/2}}{3}\right]_0^1 d\theta = \int_{\theta=0}^{\pi/2}\frac{1}{3}\,d\theta = \frac{1}{3}\times\frac{\pi}{2} = \frac{\pi}{6}.$$

You will find more surface integral examples on Exercise Sheet 10. Note that we can perform integrals over surfaces that are open (as in the above example) or closed (for example, the surface of a whole sphere). In the Divergence Theorem, however, we have a surface integral of a function of the form

$$G(x,y,z) = \mathbf{F}\cdot\hat{\mathbf{n}},$$

where the surface S is always **closed**.

6.4. The Divergence Theorem

We now have all the ingredients we need to evaluate both integrals in the Divergence Theorem.

Example. Show that the Divergence Theorem

$$\int\int\int_V \nabla\cdot\mathbf{F}\,dV = \int\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

is satisfied for the vector field $\mathbf{F} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ when V is the unit sphere.

First, we evaluate the left-hand side. Clearly, $\nabla\cdot\mathbf{F} = 3$. Hence,

$$\int\int\int_V \nabla\cdot\mathbf{F}dV = 3\int\int\int_V 1\,dV,$$

which is three times the volume of the sphere. This is $3\,(4\pi/3) = 4\pi$.

For the integral on the right, we know that $\hat{\mathbf{n}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (the radial direction vector). So,

$$\mathbf{F}\cdot\hat{\mathbf{n}} = x^2 + y^2 + z^2 = 1$$

on S. Hence,

$$\int\!\!\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \int\!\!\int_S 1\,dS,$$

which is just the surface area of the sphere. Using the standard formula for surface area, this is $4\pi(1)^2 = 4\pi$. Both integrals match.

Example. Verify that the Divergence Theorem holds when

$$\mathbf{F} = (y-x)\mathbf{i} + (y-z)\mathbf{j} + (x-y)\mathbf{k},$$

and V is the unit cube.

First we note that $\nabla\cdot\mathbf{F} = -1+1+0 = 0$. Hence $\int\int\int_V \nabla\cdot\mathbf{F}\,dV = 0$. It is not easy to describe the surface S of the unit cube, or the normal vector to S, with a single equation. In this case, it is best to split S into 6 distinct parts (the 6 faces of the cube). That is,

$$\int\!\!\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \sum_{i=1}^{6}\int\!\!\int_{S_i} \mathbf{F}\cdot\hat{\mathbf{n}}_i\,dS_i,$$

where S_i is the ith face, and $\hat{\mathbf{n}}_i$ is the unit normal vector to that face (pointing out of the cube). We evaluate these surface integrals one at a time. For example, suppose that S_1 is the top face. Then, $z = 1$ on S_1 and $\hat{\mathbf{n}}_1 = \mathbf{k}$ so $\mathbf{F}\cdot\hat{\mathbf{n}}_1 = x - y$. Now, since S_1 is flat,

$$\int\!\!\int_{S_1} \mathbf{F}\cdot\hat{\mathbf{n}}_1\,dS_1 = \int_{y=0}^{1}\int_{x=0}^{1} x - y\,dxdy = 0.$$

Similarly for the other faces .... See Exercise Sheet 10.

6.5. Line Integrals and Stokes Theorem

You studied line integrals (for lines in the x-y plane) in first year calculus. If you have forgotten, please revise your notes now. See also the **handout: Revision on line integrals in the plane**.

Consider the integral of a scalar function f along a line segment c in three space dimensions, where c starts at point a and finishes at point b. We write

$$\int_c f(x,y,z)\,ds.$$

Here, s is the 'arc length parameter'. We can use this coordinate to parametrise any line. At point a, $s = 0$ and at point b, the value of s is the length of the line segment joining a and b. If the coordinates x, y and z of any point on the line can be expressed easily in terms of s, then we just have a standard one dimensional integral

$$\int_{s=0}^{\text{length of line}} f(x(s),y(s),z(s))\,ds.$$

Now, let $\mathbf{F}$ be a vector field and consider

$$\int_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds,$$

where $\hat{\mathbf{t}}$ is the unit tangent vector to the curve c. This integral gives the work done by the vector field to move an object along c. If we know how to find $\hat{\mathbf{t}}$ then, given c, this is just a standard line integral, as above.

Draw any line segment (part of a curve) and mark two points, with arc length parameter s and $s+\delta s$. Let (x, y, z) be the standard Cartesian coordinates of the first point, and $(x+\delta x, y+\delta y, z+\delta z)$ be the coordinates of the second. Now draw a **straight** line joining these two points. This straight line has the vector equation

$$\delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k} = (x(s+\delta s) - x(s))\,\mathbf{i} + (y(s+\delta s) - y(s))\,\mathbf{j} + (z(s+\delta s) - z(s))\,\mathbf{k}.$$

If we divide by δs and take the limit as $\delta s \to 0$ then we obtain the vector

$$\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}.$$

This is, by definition, the unit tangent vector to the curve at (x, y, z). Hence

$$\hat{\mathbf{t}} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}.$$

Using this expression for $\hat{\mathbf{t}}$, we can also write

$$\int_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \int_c \mathbf{F} \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k} \right) ds = \int_c \mathbf{F} \cdot d\mathbf{r},$$

where

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}.$$

Some textbooks use the notation $\int_c \mathbf{F} \cdot d\mathbf{r}$ but we will stick to $\int_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds$. Recall also that when c is a closed loop we write $\oint_c$ and not $\int_c$.

Example. Let $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ and let c denote the closed curve shown in Figure 1. (Note that this curve lies in the x-y plane). Evaluate

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds.$$

Using the expression for $\hat{\mathbf{t}}$ (with $z = 0$) gives

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \oint_c (y\mathbf{i} - x\mathbf{j}) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} \right) ds = \oint_c \left(y\frac{dx}{ds} - x\frac{dy}{ds} \right) ds.$$

We break the curve up into three parts, and on each straight line segment, express x and y as functions of s. That is,

$$\oint_c \left(y\frac{dx}{ds} - x\frac{dy}{ds} \right) ds = \sum_{i=1}^{3} \oint_{c_i} \left(y(s)\frac{dx(s)}{ds} - x(s)\frac{dy(s)}{ds} \right) ds.$$

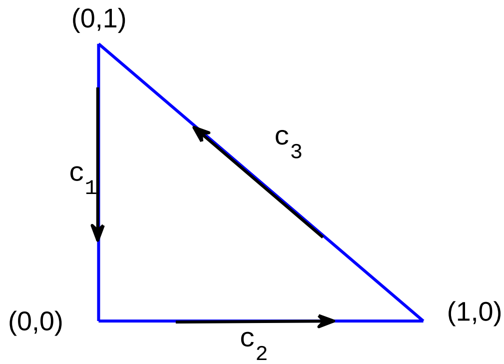

Figure 1: A closed loop corresponding to the boundary of a triangle. The arrows give the orientation of the path taken.

On c_1 (the line joining $(0,1)$ to $(0,0)$), we have $x = 0$ and $y = 1 - s$. [Check: the length of the line is 1. When $y = 1$, $s = 0$ and when $y = 0$, $s = 1$.] So,

$$\int_{c_1} \left(y(s)\frac{dx(s)}{ds} - x(s)\frac{dy(s)}{ds} \right) ds = \int_{s=0}^{1} ((1-s)0 - 0)\, ds = 0.$$

On c_2, we have $y = 0$ and $x = s$, so

$$\int_{c_2} \left(y(s)\frac{dx(s)}{ds} - x(s)\frac{dy(s)}{ds} \right) ds = \int_{s=0}^{1} (0 - s0)\, ds = 0.$$

Finally, on c_3, we have $x = 1 - s/\sqrt{2}$, and $y = s/\sqrt{2}$ so

$$\int_{c_3} \left(y(s)\frac{dx(s)}{ds} - x(s)\frac{dy(s)}{ds} \right) ds = \int_{s=0}^{1} \left(s/\sqrt{2}\right)\left(-1/\sqrt{2}\right) - \left(1 - s/\sqrt{2}\right)\left(1/\sqrt{2}\right) ds = -1.$$

Now, we have an important theorem that connects: the curl of a vector field, an open surface, and the closed curve c that 'spans' the surface. For a flat surface S in two dimensions, the closed curve c is just the boundary of S (as in the triangle example above). In three dimensions, an open surface may have curvature. Consider, for example, the upper half of the surface of the unit sphere. This surface is spanned by the closed loop corresponding to the set of points $x^2 + y^2 = 1$.

Theorem: Stokes Theorem. Let c be a closed curve and let S be an open surface spanned by c. Let $\hat{\mathbf{n}}$ be the unit normal vector to S (oriented with respect to the right-hand rule). Then, if $\mathbf{F}$ is a differentiable vector field,

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS.$$

Note that it is important here that the vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ are oriented correctly with respect to one another. In words, the theorem says that the integral of $\mathbf{F} \cdot \hat{\mathbf{t}}$ around a closed loop c is equal to the integral of the normal component of the curl of $\mathbf{F}$ across the surface S spanned by c.

Example. Let us return to the above example with $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$, where S is a right-angled triangle and c is the boundary. We have already computed

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds,$$

where the tangent vector points in the direction shown in Figure 1. Let $\hat{\mathbf{n}}$ be the unit normal vector to S pointing up (in the positive z direction). That is, $\hat{\mathbf{n}} = \mathbf{k}$. Then, since

$$\nabla \times \mathbf{F} = -2\mathbf{k},$$

we have

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = -2 \times \int\int_S 1\, dS,$$

which is minus two times the area of the triangle. Hence,

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = -2 \times \frac{1}{2} = -1.$$

This integral matches the line integral computed above. We see that Stokes Theorem is indeed satisfied.

When S is a flat surface, lying in the x-y plane, as in the above example, the computation is relatively straight forward. Exercise Sheet 10 has more challenging examples with curved surfaces. In the next example S is still a triangle but is not lying in the x-y plane.

Example. Verify Stokes Theorem when $\mathbf{F} = \left(y + y^2\right) \mathbf{k}$ and c is the boundary of the triangle S with vertices $(0, 0, 1)$, $(1, 0, 0)$, $(0, 1, 0)$.

First we compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y + y^2 \end{vmatrix} = (1 + 2y)\mathbf{i}.$$

The surface here is a portion of the plane $x + y + z = 1$ or equivalently, a portion of the surface $z = f(x, y)$ where $f(x, y) = 1 - x - y$. Using the general formula for finding a unit vector that is normal to this surface gives

$$\hat{\mathbf{n}} = \frac{-(-1)\mathbf{i} - (-1)\mathbf{j} + \mathbf{k}}{\sqrt{3}} = \frac{1}{\sqrt{3}} \left(\mathbf{i} + \mathbf{j} + \mathbf{k}\right).$$

The integral on the left-hand side of the theorem is

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = \frac{1}{\sqrt{3}} \int\int_S 1 + 2y\, dS.$$

We can convert this into a standard double integral in the x-y plane as follows.

$$\frac{1}{\sqrt{3}} \int\int_S 1 + 2y\, dS = \frac{1}{\sqrt{3}} \int_y\int_x (1 + 2y)\sqrt{3}\, dxdy.$$

We need the correct limits for x and y. The projection of S onto the x-y plane is a triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ so

$$\frac{1}{\sqrt{3}} \int\int_S 1 + 2y\, dS = \int_{y=0}^{1} \int_{x=0}^{1-y} 1 + 2y\ dxdy = \frac{5}{6}.$$

Next, we compute the line integral on the right-hand side. Note that the normal vector chosen points in the upward direction. The tangent vector $\hat{\mathbf{t}}$ to the boundary of the triangle should be oriented accordingly. (We must travel in an anti-clockwise direction.) First, note that

$$\mathbf{F} \cdot \hat{\mathbf{t}} = \left(y + y^2\right) \mathbf{k} \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) = \left(y + y^2\right) \frac{dz}{ds}$$

and so

$$\oint_c \mathbf{F} \cdot \hat{\mathbf{t}}\, ds = \oint_c \left(y + y^2\right) \frac{dz}{ds}\, ds.$$

We break the boundary up into three parts, making sure to travel in the right direction. Let c_1 denote the line segment from $(0,0,1)$ to $(1,0,0)$, let c_2 denote the line segment from and $(1,0,0)$ to $(0,1,0)$, and let c_3 denote the line segment from and $(0,1,0)$ to $(0,0,1)$.

On c_1, $s = 0$ at $(0,0,1)$ and $s = \sqrt{2}$ at $(1,0,0)$. Along this line, $y = 0$ so

$$\int_{c_1} \left(y + y^2\right) \frac{dz}{ds}\, ds = \int_{s=0}^{\sqrt{2}} 0\, \frac{dz(s)}{ds}\, ds = 0.$$

On c_2, $s = 0$ at $(1,0,0)$ and $s = \sqrt{2}$ at $(0,1,0)$. Along this line, $z = 0$ so

$$\int_{c_2} \left(y + y^2\right) \frac{dz}{ds}\, ds = \int_{s=0}^{\sqrt{2}} (y + y^2)\, 0\, ds = 0.$$

Finally, on c_3, $s = 0$ at $(0,1,0)$ and $s = \sqrt{2}$ at $(0,0,1)$. Along this line, $y = 1 - s/\sqrt{2}$ and $z = s/\sqrt{2}$ so

$$\begin{aligned} \int_{c_3} \left(y + y^2\right) \frac{dz}{ds}\, ds &= \int_{s=0}^{\sqrt{2}} \left[\left(1 - s/\sqrt{2}\right) + \left(1 - s/\sqrt{2}\right)^2\right] \left(1/\sqrt{2}\right)\, ds \\ &= \frac{1}{\sqrt{2}} \int_{s=0}^{\sqrt{2}} 2 - \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\, ds = \frac{5}{6}. \end{aligned}$$

We see that both integrals match.