Few PDEs have *exact closed-form* solutions. Even if such solutions do exist, they are often too expensive to find or to use in practical situations. In real life applications (for example, weather forecasting, aerospace engineering), scientists use computational methods to *approximate* the solutions. We will look at one such family of methods, called finite difference methods, in the next section. Separation of Variables can only be used to find the exact solution to second-order, linear, homogeneous PDEs (like the heat equation, the wave equation, and Laplace's equation) with linear, homogeneous boundary conditions. The method has several steps and is best learned by solving specific examples (see below). The general ideas are also summarised on the **handout on separation of variables.**

4.1. The One-Dimensional Heat Equation

Consider the following problem. Find $u(x,t)$ satisfying

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,$$

together with the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0,$$

and the initial condition

$$u(x,0) = u_0(x).$$

The first thing to note is that we are interested in non-zero (non-trivial) solutions. We look for 'separated solutions' of the form

$$u(x,t) = X(x)T(t)$$

where X is a function of x only, and T is a function of t only. Substituting this expression into the PDE gives

$$\frac{\partial(XT)}{\partial t} = K\frac{\partial^2(XT)}{\partial x^2},$$

and since X does not depend on t and T does not depend on x, this becomes

$$X\frac{dT}{dt} = KT\frac{d^2X}{dx^2}.$$

Notice that we no longer have partial derivatives. The next step is to re-arrange the equation so that everything that depends on x is on one side, and everything that depends on t is on the other. We have

$$\frac{1}{KT}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}.$$

Note that it is ok to divide by X and T: since we want non-zero solutions, $u = XT$ is not the zero function, so X and T are not the zero function. We have a choice about which side of the equation to put the constant K. Here, we'll put it on the t side but it doesn't really matter.

Now, notice that the left-hand side of the above equation is a **function of t only**, and the right-hand side is a **function of x only**. Since x and t are independent variables, the two sides of the equation can only be equal if they are both equal to a constant. We don't know this constant (yet) but we'll call it λ (the separation constant). Hence

$$\frac{T'}{KT} = \frac{X''}{X} = \lambda.$$

(We are now using dash notation as short-hand for the derivatives.) Hence,

$$T' - \lambda KT = 0 \quad \text{and} \quad X'' - \lambda X = 0.$$

The PDE has been transformed into two decoupled ODEs![1]

If we knew λ, then we would know the general form of the solutions $X(x)$ and $T(t)$ to our two ODEs. Unfortunately, we don't know λ so we have to consider three different cases: $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. Any (or all) of these might be possible. We check them all. First, consider the PDE for X. We can only find solutions if we have boundary conditions for X. We know that $u = XT$ satisfies the original boundary conditions, so

$$u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

(since $T(t)$ is not the zero function). Similarly,

$$u(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0.$$

We now have two boundary conditions for X and we need to solve

$$X''(x) - \lambda X(x) = 0, \quad 0 < x < L, \quad \text{such that} \quad X(0) = 0 = X(L).$$

The above equation is called an *eigenvalue problem.* To solve it, we need to find both $X(x)$ and λ. Non-zero solutions $X(x)$ are called *eigenfunctions* and the corresponding values λ are called *eigenvalues.* We test possible values for λ to see whether for those values, non-zero solutions $X(x)$ exist.

$\lambda = 0$ If $\lambda = 0$ then the ODE becomes $X''(x) = 0$ and the general solution is $X(x) = Ax + B$. The boundary condition $X(0) = 0$ tells us that $B = 0$ and the condition $X(L) = 0$ gives $AL = 0$. Since $L \neq 0$, we have $A = 0$. Since $A = B = 0$, $X(x) = 0$. Hence, **there are no non-zero solutions** when $\lambda = 0$.

$\lambda > 0$ If $\lambda > 0$ then we can write $\lambda = \omega^2$ for some $\omega \in \mathbb{R}$ with $\omega \neq 0$. The general solution to the ODE in this case[2] is

$$X(x) = Ae^{\omega x} + Be^{-\omega x}.$$

Imposing the boundary conditions $X(0) = 0$ and $X(L) = 0$ once again gives $A = B = 0$ so **there are no non-zero solutions** when $\lambda > 0$.

$\lambda < 0$ If $\lambda < 0$ then we can write $\lambda = -\omega^2$ for some $\omega \in \mathbb{R}$ with $\omega \neq 0$. The general solution in this case is

$$X(x) = A\cos(\omega x) + B\sin(\omega x).$$

[1] In first year calculus courses you studied methods for solving ODEs. It is important now that you remember these methods. If not, please revise your notes.

[2] Perhaps you remember that the characteristic or auxiliary equation is $m^2 - \lambda = 0$. To determine the general form of the solution, we examine the roots of this equation.

The condition $X(0) = 0$ gives $A = 0$. The condition $X(L) = 0$ then gives $B\sin(\omega L) = 0$. If $B = 0$ then we only have the zero solution again. Non-zero solutions can exist when $\sin(\omega L) = 0$. In other words, when $\omega L = n\pi$ for any $n \in \mathbb{Z} \setminus \{0\}$. We don't include zero because we know that $\omega L \neq 0$. The eigenvalues are therefore

$$\lambda = -\omega^2 = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

Note that we don't need to include negative values of n now because we only want to record all the distinct eigenvalues. To emphasis that there are infinitely many eigenvalues, one for each value of n, we can also write

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, \ldots$$

The corresponding non-zero solution X_n associated with λ_n is

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots$$

Here B_n is any non-zero constant. However, when we talk about 'eigenfunctions' we usually just set these constants to one and write

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots$$

So far, we have solved the ODE for $X(x)$ and found the corresponding values of λ but we have not yet solved the second ODE for $T(t)$. The good news is that we don't need to test λ again. We've already found the values λ_n that give non-zero solutions $X_n(x)$. We have one solution $T_n(x)$ for each λ_n satisfying

$$T_n'(t) - \lambda_n K T_n(t) = 0.$$

This is a first order ODE with general solution[3]

$$T_n(t) = C_n e^{K\lambda_n t}, \quad n = 1, 2, \ldots$$

(where C_n is an arbitrary constant). Putting everything together, we have found infinitely many separated solutions to the heat equation of the form

$$u_n(x,t) = X_n(x)T_n(t) = A_n e^{-K(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots,$$

for arbitrary constants A_n. (Note that when we multiply two arbitrary constants B_n and C_n, we just get another arbitrary constant.)

So, have we solved the problem? Not quite. Each function $u_n(x,t)$ satisfies the PDE and also the boundary conditions but we have not yet used the initial condition. In general, none of the functions $u_n(x,t)$ will satisfy the initial condition $u_n(x,0) = u_0(x)$ (unless $u_0(x)$ is a very special function). To find a solution that does satisfy the initial condition, the final step is to apply the Principle of Superposition. Since the PDE and the boundary conditions are linear and homogeneous, and the functions $u_n(x,t)$ are linearly independent[4], we know that any linear combination is also a solution. Consider

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-K(n\pi/L)^2 t} \sin\left(\frac{n\pi x}{L}\right). \tag{1}$$

[3] You know have seen this in first year. Please recall the method of integrating factors, if it is not obvious.

[4] We haven't actually proved this but the eigenfunctions are always linearly independent. Can you prove it?

This is also a solution to the heat equation and it satisfies zero boundary conditions. We can make it satisfy the initial condition if we can find coefficients A_n so that

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = u_0(x).$$

In other words, if the function $u_0(x)$ can be written as a Fourier Sine Series, with Fourier coefficients A_n! Using the orthogonality property of the eigenfunctions (i.e., the sine functions), we know that the Fourier coefficients are

$$A_n = \frac{\int_0^L u_0(x)\sin(n\pi x/L)\,dx}{\int_0^L \sin^2(n\pi x/L)\,dx} = \frac{2}{L}\int_0^L u_0(x)\sin(n\pi x/L)\,dx. \tag{2}$$

Hence, the final solution is the infinite series $u(x,t)$ given in (1) with coefficients defined in (2).

Summary. It is a good idea to remember the **steps** of the method. Do not try and memorize the answer to the above problem! The boundary conditions are key. They determine the eigenvalues λ_n and the eigenfunctions $X_n(x)$. Changing just one boundary conditions leads to a completely different solution. Try question 4 on Exercise Sheet 5 and compare your answer with the above calculation.

4.2. The One-Dimensional Wave Equation

Consider the following problem. Find $u(x,t)$ satisfying

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \quad t > 0,$$

such that

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad \frac{\partial u(L,t)}{\partial x} = 0,$$

and

$$u(x,0) = u_0(x), \qquad \frac{\partial u(x,0)}{\partial t} = u_1(x).$$

We can apply the method of Separation of Variables exactly as we did above for the heat equation but there are two key differences:

- two initial conditions need to be imposed after we find all the separated solutions,
- we obtain a second-order ODE for $T(t)$.

You are asked to find solutions to the wave equation in questions 1 and 2 on Exercise Sheet 6, so full details are not given here. To start off, we set $u(x,t) = X(x)T(t)$ and substitute this into the PDE to obtain

$$XT'' = c^2TX'',$$

or

$$\frac{T''}{c^2T} = \frac{X''}{X} = \lambda,$$

where λ is the separation constant. We obtain two second-order ODEs:

$$X'' - \lambda X = 0, \qquad T'' - c^2\lambda T = 0.$$

The ODE for $X(x)$ is the same as before, but now the boundary conditions are different. The first task is to use the boundary conditions for u to obtain boundary conditions for X and then find the eigenvalues λ and the corresponding eigenfunctions (non-zero solutions) X. Once we know λ, we can then also solve the ODE for $T(t)$.

4.3. Laplace's Equation in Two Dimensions

Consider the following problem. Find $u(x, y)$ satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < a, \quad 0 < y < b.$$

Can we use Separation of Variables? Following the same steps in the examples above, we can start by substituting $u(x, y) = X(x)Y(y)$ into the PDE to give $YX'' + XY'' = 0$. Rearranging gives

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda,$$

where λ is the separation constant. We obtain two ODEs

$$X'' - \lambda X = 0, \qquad Y'' + \lambda Y = 0.$$

Note the difference in the sign! To make any progress, we'll need boundary conditions for both $X(x)$ and $Y(y)$.

Earlier, we said that 'Separation of Variables can only be used for problems with homogeneous (i.e. zero) boundary conditions' but if we apply zero boundary conditions everywhere on $[0, a] \times [0, b]$ then the only solution to Laplace's equation is $u(x, y) = 0$. In fact, what we really need is the property that when we add up all the separated solutions, the series solution should still satisfy the boundary conditions. If the boundary conditions are homogeneous (i.e., zero), then adding up infinitely many functions that are zero on the boundary still produces something that is zero on the boundary. If we add up infinitely many separated solutions that are equal to say, one on the boundary, then the sum of the solutions is not equal to one on the boundary. In fact, the sum blows up. For Laplace's equation on a rectangle, we can have zero boundary conditions on three sides and a non-zero condition on the fourth side. We treat the non-zero condition like an 'initial condition' in the sense that we only impose it only on the series solution, not on the individual separated solutions.

Consider the boundary conditions

$$u(0, y) = 0, \quad u(a, y) = 0, \quad u(x, 0) = 0, \quad u(x, b) = f(x).$$

Substituting $u(x, y) = X(x)Y(y)$ in the first three of these conditions gives $X(0) = 0, X(a) = 0$ and $Y(0) = 0$. Since we have a full set of boundary conditions for the ODE $X'' - \lambda X = 0$, we solve that one first. We've solved this ODE before (in the heat equation example) but it doesn't hurt to review it again here. We test the possible cases for λ as follows:

$\lambda = 0$ If $\lambda = 0$ then the ODE becomes $X''(x) = 0$ and the general solution is $X(x) = Ax + B$. The boundary conditions $X(0) = 0$ and $X(a) = 0$ tell us that **there are no non-zero solutions** in this case.

$\lambda > 0$ If $\lambda > 0$ then we can write $\lambda = \omega^2$ for some $\omega \in \mathbb{R}$ with $\omega \neq 0$. The general solution to the ODE in this case is

$$X(x) = Ae^{\omega x} + Be^{-\omega x}.$$

Imposing the boundary conditions $X(0) = 0$ and $X(a) = 0$ reveals that there are also **no non-zero solutions.**

$\lambda < 0$ If $\lambda < 0$ then we can write $\lambda = -\omega^2$ for some $\omega \in \mathbb{R}$ with $\omega \neq 0$. The general solution in this case is

$$X(x) = A\cos(\omega x) + B\sin(\omega x).$$

The condition $X(0) = 0$ gives $A = 0$. The condition $X(a) = 0$ then gives $B\sin(\omega a) = 0$. If $B = 0$ then we only have the zero solution again. Non-zero solutions exist when $\sin(\omega a) = 0$. In other words, when $\omega a = n\pi$ for any $n \in \mathbb{Z} \setminus \{0\}$ (since $\omega L \neq 0$). The eigenvalues are therefore

$$\lambda_n = -\omega^2 = -\left(\frac{n\pi}{a}\right)^2, \quad n = 1, 2, \ldots$$

The non-zero solution X_n associated with λ_n is

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right), \quad n = 1, 2, \ldots$$

where B_n is an arbitrary constant.

Next, we need to solve

$$Y_n''(y) + \lambda_n Y_n(y) = 0, \quad \text{such that} \quad Y_n(0) = 0,$$

where we know that $\lambda_n < 0$. In this case the general form of the solution is

$$Y_n(y) = A_n e^{\omega_n y} + B_n e^{-\omega_n y}.$$

The boundary condition $Y_n(0) = 0$ tells us that $A_n + B_n = 0$ or $B_n = -A_n$. Hence,

$$Y_n(y) = A_n\left(e^{\omega_n y} - e^{-\omega_n y}\right) = 2A_n \sinh\left(\omega_n y\right) = 2A_n \sinh\left(\frac{n\pi y}{a}\right).$$

We have found infinitely many separated solutions

$$u_n(x, y) = X_n(x)Y_n(y) = A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right), \quad n = 1, 2, \ldots.$$

(The product of two arbitrary constants is just an arbitrary constant). None of these individual solutions satisfies the fourth boundary condition. Applying the principle of superposition,

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi y}{a}\right)$$

is also a solution to the the PDE, and satisfies the three zero boundary conditions. To impose the final non-zero boundary condition, we need to find coefficients A_n such that

$$f(x) = u(x, b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi b}{a}\right).$$

Using the orthogonality property of the sine functions (on the interval $[0, a]$), we have

$$A_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\,dx}{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)\,dx}.$$

Calculating the integral on the denominator and rearranging gives

$$A_n = \left(\frac{2}{a}\right)\frac{1}{\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\,dx.$$

4.4. The Wave Equation on a Disk

Consider a circular drum (i.e., a thin circular membrane, fixed to a rigid circular frame) of radius a. We can solve the wave equation to model vibrations in the membrane, given its initial shape, for example at $t = 0$. Recall, the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

(where we have set $c^2 = 1$ for simplicity). In two space dimensions, this is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

On circular (two-dimensional) geometries it is best to work in polar coordinates. So, using the chain rule for partial derivatives, we can transform this into

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

To simplify things a bit, let's assume that

- the vibrations are radially symmetric (do not depend on θ)
- the interesting vibrations correspond to negative values of the separation constant only (the case $\lambda < 0$ or $\lambda = -\omega^2$).

The first assumption means that u is a function of t and r only and we need to solve

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}, \qquad 0 < r < a, \quad t > 0.$$

This reduced problem is one-dimensional in space. (Starting at the centre of the drum and drawing a straight line to the boundary, we've effectively taken a one-dimensional slice). We'll need two boundary conditions, one at $r = 0$ and one at $r = a$. If the membrane is fixed to the frame, then the vibration is zero there. The correct boundary condition is

$$u(a, t) = 0.$$

At $r = 0$ we will impose the condition

$$| \, u(0, t) \, | < \infty.$$

This simply says that the size of the vibrations at the centre of the drum are finite.

To apply separation of variables, we substitute $u(r, t) = R(r)T(t)$ into the PDE to obtain two ODEs

$$r^2 R''(r) + rR'(r) - \lambda r^2 R(r) = 0,$$

and

$$T''(t) - \lambda T(t) = 0.$$

Substituting into the boundary conditions gives

$$R(a) = 0, \qquad | \, R(0) \, | < \infty.$$

Since we have two boundary conditions for R, we can solve the ODE for $R(r)$ first, using the assumption that $\lambda = -\omega^2$ for some $\omega \in \mathbb{R}$.

The above ODE for $R(r)$ is perhaps one that you have not met before. It is called Bessel's equation of order zero. The general form of the solution is

$$R(r) = AJ_0(\omega r) + BY_0(\omega r),$$

where A, B are arbitrary constants. The functions Y_0 and J_0 are special functions called Bessel functions. More details about these functions, and how to investigate them in MATLAB are available on the **handout on Bessel functions.** A key point is that $J_0(0)$ is finite but Y_0 blows up at the origin. So, to satisfy the boundary condition at $r = 0$, we must have $B = 0$. The boundary conditions at $r = a$ then gives

$$0 = R(r) = AJ_0(\omega a).$$

If A is zero, then we only have $R(r) = 0$. Non-zero solutions exist when $J_0(\omega a) = 0$. That is, when ωa is a zero of the function J_0. In fact, J_0 has infinitely many zeros (just like sine and cosine). Let α_n denote a zero of J_0, for $n = 1, 2, \ldots$ then we have

$$\omega_n = \frac{\alpha_n}{a}, \qquad n = 1, 2, \ldots.$$

The eigenvalues are

$$\lambda_n = -\left(\frac{\alpha_n}{a}\right)^2, \qquad n = 1, 2, \ldots,$$

and the eigenfunctions are

$$R_n(r) = A_n J_0(\alpha_n r/a), \qquad n = 1, 2, \ldots.$$

Next, we need to solve for $T_n(t)$ and combine with $R_n(r)$ to find the separated solutions $u_n(r, t) = R_n(r)T_n(t)$.