In the next section, we will study a method called *separation of variables* for finding exact solutions to a certain class of partial differential equations (PDEs). In this section, we introduce div & grad notation, for writing PDEs in a compact way, and discuss some of the properties of PDEs. PDEs are classified into different groups, according to their properties. By now, you should have read the **handout on Classical PDEs**, which contains a list of all the PDEs you will meet in this course.

Definition: PDE, solution. A partial differential equation or *PDE* is an equation containing partial derivatives of an unknown function (say, u). A *solution* is a function which, when substituted for u, satisfies the PDE.

Example. The equation

$$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0$$

is a PDE. The function $u = 6x - 3y^2$ is a solution.

A PDE (without extra conditions) will not have a unique solution. In fact, there will usually be infinitely many solutions. Conditions such as $u(x, 0) = g(x)$ (where $u = u(x, t)$ and t represents time) are called *initial conditions*. Conditions such as $u(0, t) = f(t)$ (where x represents space) are called *boundary conditions*. Unique solutions are only possible if extra conditions are supplied.

3.1. Grad & Div Notation

The symbol ∇ ('grad' or gradient) is a differential operator. When we apply it to a function, it produces the vector of first partial derivatives (in Cartesian coordinates) of that function.

- In 1d, with $u = u(x)$, we have $\nabla u = \frac{\partial u}{\partial x}$ (a one component vector).
- In 2d, with $u = u(x, y)$, we have $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$.
- In 3d, with $u = u(x, y, z)$, we have $\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$.

Notice that ∇ acts on *scalar* functions and produces a *vector*. The symbol $\nabla\cdot$ ('div' or divergence) also represents a differential operator and is used on vector functions $\mathbf{F} = (F_1, F_2, F_3)$ as follows:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

We can use the two operators together in the following way.

$$\nabla \cdot (\nabla u) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

The symbol ∇^2, or also Δ, is called the *Laplace* operator (or *Laplacian*). This is short-hand for $\nabla \cdot \nabla$. Hence, we apply it as

$$\nabla^2 u = \Delta u = \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

When working with PDEs, we use this notation as much as possible.

3.2. Classification of PDEs

To know how to solve PDEs, we first need to classify them. Not all solution methods are suitable for all types of PDEs. We use the following words to classify PDEs:

- order
- linear or nonlinear
- homogeneous or non-homogeneous
- elliptic, parabolic, or hyperbolic (for second-order PDEs).

Definition: Order. The *order* of a PDE is the order of the highest derivative appearing in the equation.

Example. The PDE

$$\frac{\partial^3 u}{\partial x^2 \partial y} + \frac{\partial^2 u}{\partial x \partial y} = x$$

has order three (or 'it is of third-order').

We can write any PDE in operator notation

$$\mathcal{L}u = f,$$

where u is the solution, $\mathcal{L}$ is a differential operator (that acts on u) and f is a collection of terms involving only the independent variables (and not u).

Example. The PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2x = 0$$

can be written as $\mathcal{L}u = f$ with

$$\mathcal{L} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad f = 2x.$$

Definition: Linear. The operator $\mathcal{L}$ is *linear* if, for any two functions u_1, u_2, and any $c \in \mathbb{R}$,

1. $\mathcal{L}(u_1 + u_2) = \mathcal{L}u_1 + \mathcal{L}u_2$,
2. $\mathcal{L}(cu_1) = c\mathcal{L}u_1$.

If $\mathcal{L}$ is linear, then the PDE $\mathcal{L}u = f$ is *linear*. Otherwise, it is *nonlinear*.

Example. The PDE

$$\frac{\partial u}{\partial x} + \sin(u) = 0$$

is nonlinear because

$$\mathcal{L}(u_1 + u_2) = \frac{\partial (u_1 + u_2)}{\partial x} + \sin(u_1 + u_2) \neq \frac{\partial u_1}{\partial x} + \sin(u_1) + \frac{\partial u_2}{\partial x} + \sin(u_2) = \mathcal{L}u_1 + \mathcal{L}u_2.$$

However, the PDE $\frac{\partial u}{\partial x} + \sin(x) = 0$ is linear!

Basically, linear PDEs can have no terms in them in which the solution or its derivatives have powers greater than one, no products of derivatives, logs, trigonometric functions, etc.

Definition: Homogeneous. A PDE of the form $\mathcal{L}u = 0$ is homogeneous (i.e., there are no terms that do not involve the unknown u or its derivatives).

Example. The PDE $u_{tt} = u_{xx}$ can be written as $u_{tt} - u_{xx} = 0$, or $\mathcal{L}u = 0$ where

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}.$$

Hence, it is homogeneous.

Definition: Elliptic, parabolic, hyperbolic. A linear second-order PDE for a function $u(x, t)$ (i.e., a function of two variables) can be written in the form

$$Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu + G = 0,$$

where A–F are functions of x and t only, and $G = 0$ for homogeneous problems. We say the PDE is

- elliptic if $B^2 - 4AC < 0$,
- parabolic if $B^2 - 4AC = 0$,
- hyperbolic if $B^2 - 4AC > 0$.

(Notice that this test involves only the coefficients of the second-order derivatives.)

Example. Laplace's equation in two dimensions for $u = u(x, y)$ is

$$\nabla^2 u = 0, \quad \text{or equivalently,} \quad u_{xx} + u_{yy} = 0.$$

We have $A = 1, B = 0$ and $C = 1$ so $B^2 - 4AC = -4$. Since $-4 < 0$, Laplace's equation is elliptic.

On the **handout on classical PDEs** you will also find examples of parabolic and hyperbolic PDEs. See also Exercise Sheet 4.

PDEs that are linear and homogeneous have a very important property. Let u_1 and u_2 be two linearly independent solutions to $\mathcal{L}u = 0$ and suppose that $\mathcal{L}$ is linear. Let $c_1, c_2 \in \mathbb{R}$ and define $v = c_1u_1 + c_2u_2$. Then, using the properties of $\mathcal{L}$ and the fact that u_1 and u_2 are solutions gives

$$\mathcal{L}v = \mathcal{L}\left(c_1u_1 + c_2u_2\right) = \mathcal{L}\left(c_1u_1\right) + \mathcal{L}\left(c_2u_2\right) = c_1\mathcal{L}\left(u_1\right) + c_2\mathcal{L}\left(u_2\right) = 0 + 0 = 0.$$

Hence, v is also a solution. Since c_1 and c_2 are arbitrary, there are infinitely many solutions. We can do a similar exercise if we start with three linearly independent solutions, then four, five, and so on. This leads us to the **Principle of Superposition.** This says that if we can find a set of linearly independent solutions of a linear and homogeneous PDE, then any linear combination of them is also a solution.

Theorem. Principle of Superposition. Let $\{u_1, u_2, \ldots\}$ be a set of infinitely many linearly independent solutions to $\mathcal{L}u = 0$ where $\mathcal{L}$ is linear. Then, for any $c_i \in \mathbb{R}$,

$$u = \sum_{i=1}^{\infty} c_i u_i$$

is also a solution.

The above result also says that linear and homogeneous PDEs may have solutions that are in the form of an infinite series. This should remind you of our earlier discussion of Fourier Series and indeed, we will link back to Fourier Series later.

Second-order, linear, homogeneous PDEs can be solved by the method of separation of variables. Below, we briefly discuss three model problems that belong to this class. One is elliptic, one is parabolic, and one is hyperbolic. The heat equation and the wave equation were also discussed on the **handout on classical PDEs.**

3.3. The One-Dimensional Heat Equation

Consider a one-dimensional metal wire of length L. Let u denote the temperature in the wire. We assume $u = u(x,t)$ since the temperature will depend on time and also on the position along the wire. If there are no heat sources or sinks along the wire, then heat energy moves via the process of diffusion only. The mathematical model of this physical law is given by

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2},$$

or $u_t = K\nabla^2 u$, where K is the thermal conductivity coefficient. To obtain a unique solution, we need to specify boundary conditions and one initial condition. These will depend on the physical situation we want to model. Possibilites for **boundary conditions** include:

- $u(0,t) = 0$ and $u(L,0) = 0$ (the temperature is zero at the boundary),
- $u(0,t) = 10$ and $u(L,0) = 0$ (the temperature is ten at one end of the wire, but zero at the other),
- $u_x(0,t) = 0$ and $u_x(L,t) = 0$ (the flow of heat at the ends of the wire is zero, or, the boundaries are 'insulated').

For the **initial condition**, we need to specify an initial temperature at $t = 0$. That is, $u(x,0) = f(x)$, for some $f(x)$ that also agrees with the chosen boundary conditions.

Example: Heat Equation. Consider the heat equation

$$u_t = u_{xx}, \qquad 0 < x < 1,\, t > 0,$$

with $u(0,t) = 0$ and $u(1,t) = 0$ and initial condition

$$u(x,0) = \sin(\pi x) + \sin(3\pi x).$$

The solution is

$$u(x,t) = e^{-t}\sin(\pi x) + e^{-9t}\sin(3\pi x)\,.$$

This is plotted for various values of t in Figure 1. Can you interpret what is happening physically? (Think of the different graphs as representing frames in a movie.)

3.4. The One-Dimensional Wave Equation

Consider a string, which when laid flat, has length L. Let $u(x,t)$ be the displacement of the string from the horizontal axis. Then, u satisfies the PDE

$$u_{tt} = c^2 u_{xx},$$

where c^2 is the wave speed (related to the properties of the string). If the string is fixed to the horizontal axis at both ends (at $x = 0$ and $x = L$), then there is no displacement and the boundary conditions are simply

$$u(0, t) = 0, \quad u(L, t) = 0.$$

We also need **two** initial conditions. For example,

$$u(x, 0) = f(x), \qquad u_t(x, 0) = g(x),$$

for given functions $f(x)$ and $g(x)$.

Example: Wave Equation. Consider the wave equation

$$u_{tt} = u_{xx}, \qquad 0 < x < 1,\ t > 0,$$

with $u(0, t) = 0$ and $u(1, t) = 0$ and initial conditions

$$u(x, 0) = \sin(\pi x) \quad \text{and} \quad \frac{\partial u(x, 0)}{\partial t} = 0.$$

The solution is

$$u(x, t) = \sin(\pi x) \cos(\pi t).$$

This is plotted for various values of t in Figure 2. Can you interpret what is happening physically?

3.5. Laplace's Equation in Two Dimensions

Consider the PDE $\nabla^2 u = 0$ or $u_{xx} + u_{yy} = 0$. Notice that this is the heat eqation in two dimensions with $u_t = 0$. Laplace's equation can therefore be used to model steady-state temperatures. Suppose that we want to model the temperature u in a rectangular plate represented by the region $[0, L_x] \times [0, L_y]$. Here, the temperature is a function of the two space coordinates x and y but does not depend on time t. The lengths L_x and L_y give the dimensions of the plate. To find a unique steady-state temperature, we need to supplement the PDE with four boundary conditions (one for each side of the plate), such as

$$u(x, 0) = f_1(x), \quad u(x, L_y) = f_2(x), \quad u(0, y) = g_1(y), \quad u(L_x, y) = g_2(y).$$

Example: Laplace's Equation. Consider the PDE

$$u_{xx} + u_{yy} = 0, \qquad 0 < x < \pi,\ 0 < y < 1,$$

with boundary conditions

$$u(x, 0) = 0, \quad u(x, 1) = \pi, \quad u(0, y) = 0, \quad u(\pi, y) = 0.$$

A MATLAB surf plot of the solution is shown in Figure 3. In this case, there is no neat expression we can write down for $u(x, y)$. The solution is an infinite series. We will study how to find it in the next section.

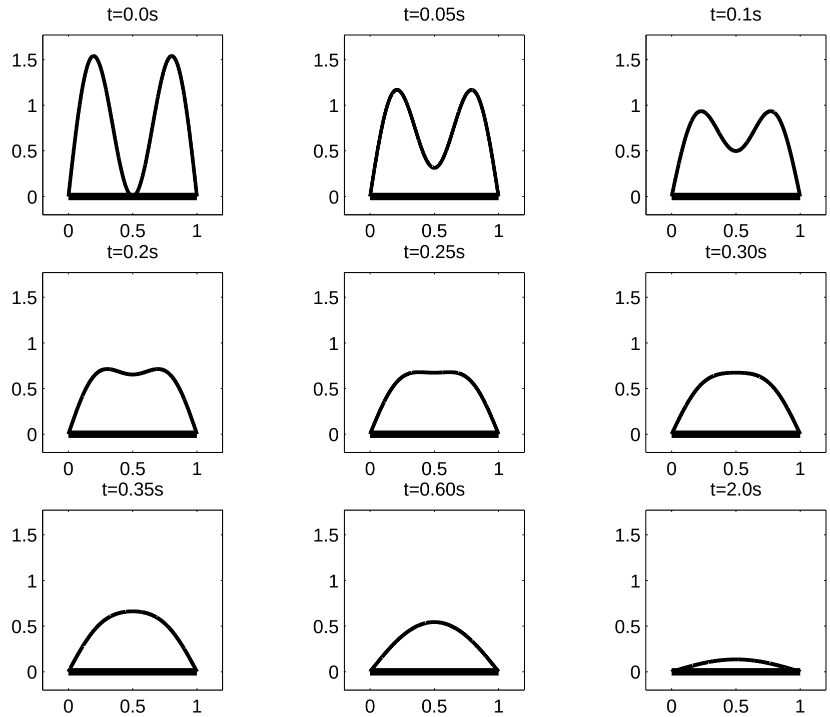

Figure 1: Snapshots of the solution $u(x,t)$ to the heat equation example at different values of t (time).

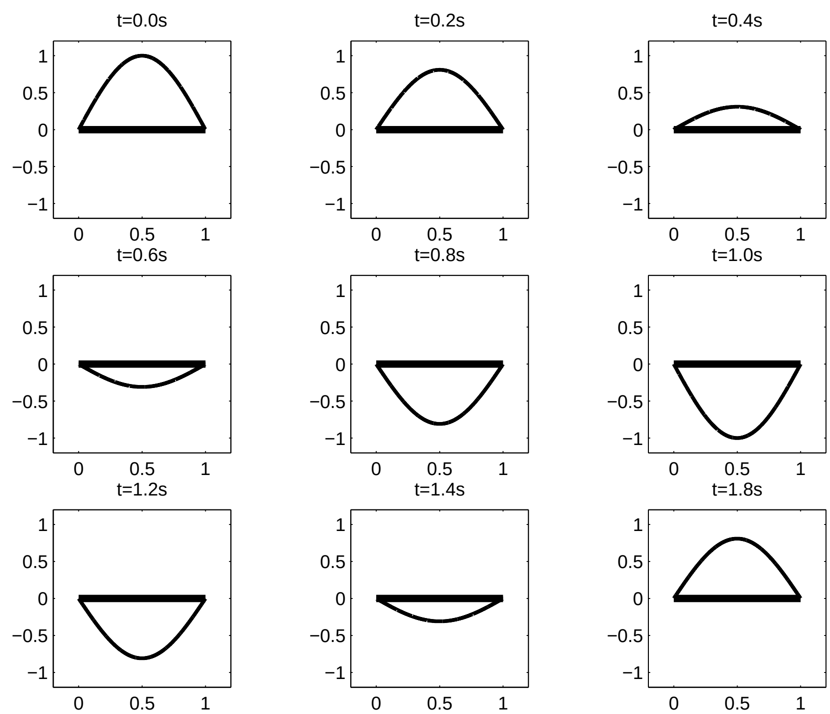

Figure 2: Snapshots of the solution $u(x,t)$ to the wave equation example at different values of t (time).

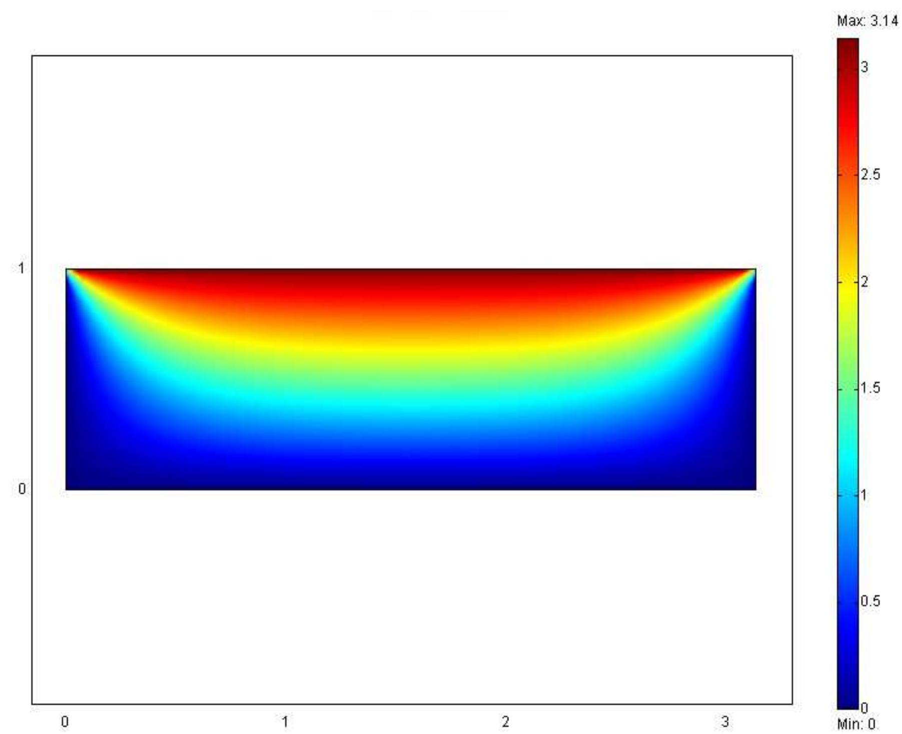

Figure 3: MATLAB `surf` plot of the solution $u(x, t)$ to Laplace's equation.