In section 4, we will study a method called *separation of variables* for finding exact solutions to a certain class of partial differential equations (PDEs). To do this, it will be necessary to express a given function of **one** variable $f(x)$ as a series

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \qquad a \leq x \leq b, \tag{1}$$

called a **Fourier series.** In a Fourier series, the coefficients c_n are expressed as integrals and the functions ϕ_n are **orthogonal**. You met orthogonal vectors in the first year Linear Algebra course. You should revise your lecture notes on orthogonality from that course, and additionally, read the **handout on orthogonal vectors.**

2.1. Orthogonal Functions

It is impossible to expand **all** functions $f(x)$ and we certainly shouldn't write $=$ in (1) until we know whether the series actually converges. However, we can expand a certain class of functions, known as **piecewise continuous** functions.

Definition: Piecewise continuous (pwc). Given a function $f(x)$ on $[a, b]$, $f(x)$ is **piecewise continuous (pwc)** on $[a, b]$ if there exists a finite number of points x_n, with

$$a = x_0 < x_1 < \cdots < x_N = b$$

such that

1. $f(x)$ is continuous on each open subinterval (x_{n-1}, x_n), and
2. $f(x)$ has a finite limit at each end of each open interval.

In summary: $f(x)$ is allowed to have breaks/jumps as long as there are only a finite number of these, and $f(x)$ does not blow up (go to infinity) anywhere in $[a, b]$.

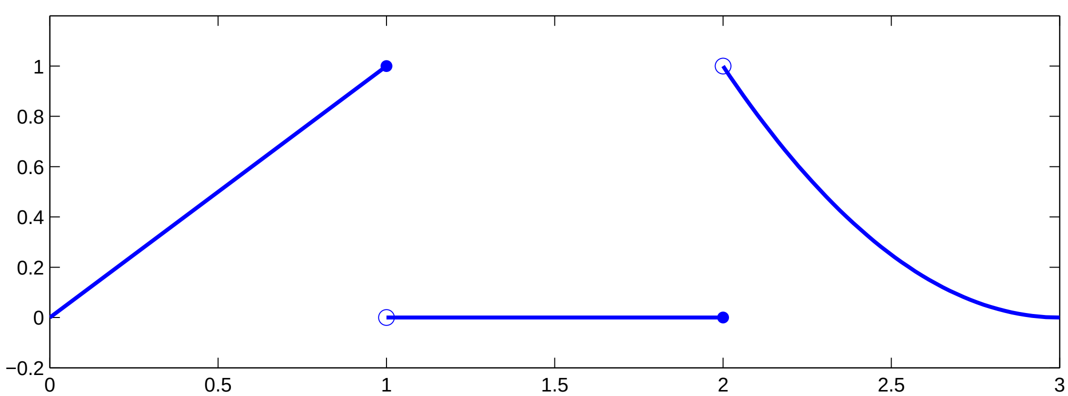


Figure 1: An example of a piecewise continuous function $f(x)$ on $[0, 3]$.

Example. Consider the function $f(x)$ shown in Figure 1. $f(x)$ is pwc on $[0,3]$ (but not continuous) since $f(x)$ is continuous on each of the open subintervals $(0,1),(1,2),(2,3)$ and the limits of $f(x)$ at the end points of these open subintervals exist (are finite). In particular, $f(0^+)=0, f(1^-)=1$, $f(1^+)=1, f(2^-)=0$, and $f(2^+)=1, f(3^-)=0$.

Now, pwc functions are Riemann integrable, which allows us to define orthogonality. Compare the following definitions for functions with those you already know for vectors.

Definition: Inner-product. Given two pwc functions $f(x), g(x)$ on $[a,b]$,

$$(f,g) := \int_a^b f(x)g(x)\,dx$$

defines an **inner-product.**

Definition: Norm. The **norm** of a pwc function $f(x)$ on $[a,b]$ is

$$\|f\| := \sqrt{(f,f)} = \left(\int_a^b f(x)^2\,dx\right)^{1/2}.$$

Definition: Orthogonal. Two pwc functions $f(x)$ and $g(x)$ on $[a,b]$ are **orthogonal** when

$$(f,g) = \int_a^b f(x)g(x)\,dx = 0$$

and **orthonormal** when, in addition to being orthogonal, we have $\|f\| = 1 = \|g\|$.

2.2. Generalised Fourier Series

Suppose $f(x)$ is pwc on $[a,b]$ and let $\{\phi_n(x)\}_{n=1}^{\infty}$ be a specified set of orthogonal pwc functions on $[a,b]$. Does it make sense to write:

$$f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x), \qquad a \le x \le b\,?$$

We can't write $=$ until we have checked if the sum **converges**. If it converges, we can work out the coefficients easily. Taking the inner-product of both sides with any of the functions $\phi_i(x)$ gives

$$\int_a^b f(x)\phi_i(x)\,dx \quad = \quad \int_a^b \left(\sum_{n=1}^{\infty} c_n\phi_n(x)\right)\phi_i(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b \phi_n(x)\phi_i(x)\,dx.$$

Since $(\phi_n,\phi_i)=0$ unless $n=i$, only one of the integrals in the sum is non-zero. Hence,

$$\int_a^b f(x)\phi_i(x)\,dx \quad = \quad c_i\int_a^b \phi_i(x)\phi_i(x)\,dx$$

and solving for c_i gives

$$c_i = \frac{\int_a^b f(x)\phi_i(x)\,dx}{\int_a^b \phi_i(x)^2\,dx} = \frac{(f,\phi_i)}{(\phi_i,\phi_i)}.$$

The series becomes

$$f(x) = \sum_{n=1}^{\infty} \frac{(f,\phi_n)}{(\phi_n,\phi_n)}\,\phi_n(x).$$

This is a **generalised Fourier series** and the coefficients c_n are called **Fourier coefficients.**

2.3. Fourier Series

Usually, when we talk about 'Fourier Series', we make a particular choice for $[a, b]$ and the set of orthogonal functions $\{\phi_n(x)\}_{n=1}^{\infty}$.

Consider the following integrals on $[a, b] = [-L, L]$, $L > 0$:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\,dx, \quad \text{where } n, m = 1, 2, \ldots.$$

Simplifying the integrand gives

$$\begin{aligned} & \frac{1}{2}\int_{-L}^{L} \sin\left(\frac{(n+m)\pi x}{L}\right) + \sin\left(\frac{(n-m)\pi x}{L}\right)\,dx \\ = \;& \frac{1}{2}\left[-\frac{L}{(n+m)\pi}\cos\left(\frac{(n+m)\pi x}{L}\right) - \frac{L}{(n-m)\pi}\cos\left(\frac{(n-m)\pi x}{L}\right)\right]_{-L}^{L} = 0. \end{aligned}$$

This tells us that the set of functions $\{\sin(\pi x/L), \sin(2\pi x/L), \ldots\}$ and the set $\{\cos(\pi x/L), \cos(2\pi x/L), \ldots\}$ are mutually orthogonal on the interval $[-L, L]$.

We can also show (see Exercise Sheet 2) that

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}$$

and

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}.$$

With a little more work (see Exercise Sheet 2), it can be shown that the set

$$\{1, \sin(\pi x/L), \cos(\pi x/L), \sin(2\pi x/L), \cos(2\pi x/L), \ldots\}$$

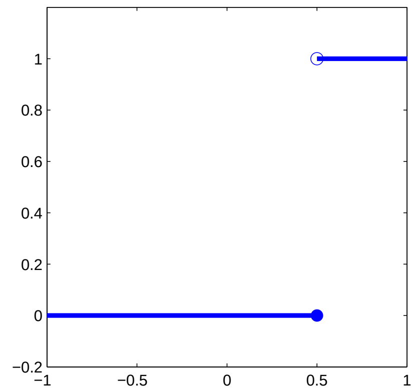
is an **orthogonal** set of functions on $[-L, L]$. This is the set we will use for $\{\phi_n(x)\}_{n=1}^{\infty}$.

Definition: Fourier Series. The Fourier series of a function $f(x)$ that is pwc on $[-L, L]$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

with Fourier coefficients

$$\begin{aligned} a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx, \quad n = 1, 2, \ldots, \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx, \quad n = 1, 2, \ldots. \end{aligned}$$



Example. Compute the Fourier Series of the following pwc function

$$f(x) = \begin{cases} 0 & -1 \le x \le 1/2 \\ 1 & 1/2 < x \le 1 \end{cases} .$$

Direct integration gives:

$$a_0 = \frac{1}{2} \int_{-1}^{1} f(x)\, dx = \frac{1}{2} \int_{1/2}^{1} 1\, dx = \frac{1}{4}$$

(note that $L = 1$ here). Similarly,

$$a_n = \frac{1}{1} \int_{-1}^{1} f(x) \cos(n\pi x)\ dx = \int_{1/2}^{1} \cos(n\pi x)\ dx = -\frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right),$$

and

$$\begin{aligned} b_n &= \frac{1}{1} \int_{-1}^{1} f(x) \sin(n\pi x)\ dx = \int_{1/2}^{1} \sin(n\pi x)\ dx \\ &= \left[-\frac{1}{n\pi} \cos(n\pi x) \right]_{1/2}^{1} = -\frac{1}{n\pi} \cos(n\pi) + \frac{1}{n\pi} \cos\left(\frac{n\pi}{2}\right) \\ &= \frac{1}{n\pi} \left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n \right). \end{aligned}$$

Putting all this together, the Fourier series associated with $f(x)$ is

$$\frac{1}{4} + \sum_{n=1}^{\infty} \left(-\frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right) \cos(n\pi x) + \sum_{n=1}^{\infty} \frac{1}{n\pi} \left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n \right) \sin(n\pi x).$$

Notice that we did *not* say "$f(x)$ is *equal* to the Fourier series." A natural question now is: to what function does the Fourier series converge? Does it converge to $f(x)$ (the function in the figure above)? This is investigated on a separate **handout on Fourier Series**, which you should now read.

The numerical investigation in MATLAB described in the handout reveals that the Fourier series is not equal to $f(x)$ everywhere. We observe that:

1. The Fourier series agrees with $f(x)$ on $[-1,1]$ (i.e., converges to $f(x)$) **except** at three points: $x=-1$, $x=0.5$ and $x=1$. At those points, the Fourier series converges to the value $1/2$.

2. If we plot the Fourier series on $(-\infty,\infty)$ we just see copies of the series on $[-1,1]$, shifted. The Fourier series is a *periodic* function.

Definition: Periodic function. A function $f(x)$ is periodic with period T if, for all x, $f(x+T)=f(x)$.

Examples: $\sin(3x),\ \sin(4\pi x),\ \tan(x),\ldots$ are periodic functions.

Given a function $f(x)$ defined on a fixed finite interval, we can always make a 'periodic version' of it, by extending it to the whole real number line.

Definition: Periodic extension. Let $f(x)$ be defined on $[-L,L]$. The periodic extension $\tilde{f}(x)$ of $f(x)$ is defined by

$$\tilde{f}(x)=\begin{cases} f(x) & -L\le x<L \\ \tilde{f}(x-2L) & x\ge L \\ \tilde{f}(x+2L) & x<-L \end{cases}.$$

It is easier to draw a picture. Basically, we take $f(x)$ on $[-L,L]$ and copy it on the adjacent intervals of length $2L$, taking care not to give $f(x)$ duplicate values at the end points of any of these subintervals. Of course, we cannot sketch $\tilde{f}(x)$ on the whole of $(-\infty,\infty)$, but it is best to include at least the intervals to the right and left of the principle interval: $[-3L,-L]$, $[-L,L]$ and $[L,3L]$.

Example. Sketch the periodic extension of the function

$$f(x)=\begin{cases} 0 & -1\le x\le 1/2 \\ 1 & 1/2<x\le 1 \end{cases}.$$

Using the definition, we copy the definition of $f(x)$ on $[-1,1)$ (not including the right end point) and then on the interval $[1,3)$ we copy the definition of $f(x)$ on $[-1,1)$. Similarly, on the interval $[-3,-1)$ we copy the definition of $f(x)$ on $[-1,1)$.

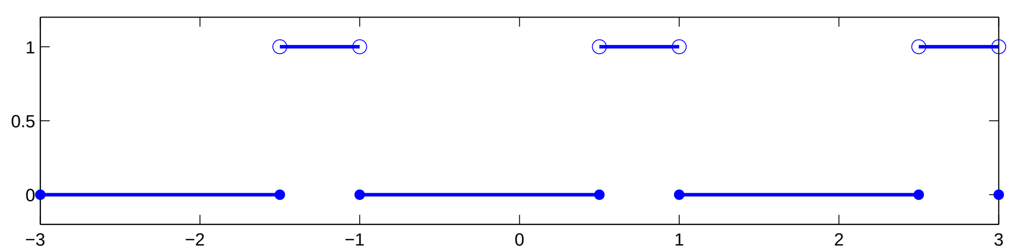


Figure 2: Periodic extension $\tilde{f}(x)$ of the function $f(x)$ defined on the fixed interval $[-1,1]$.

Looking at the handout, we see that the Fourier series associated with $f(x)$ 'looks like' the periodic extension $\tilde{f}(x)$, except at the points where $\tilde{f}(x)$ jumps. This is not a coincidence. Fourier's Theorem explains the connection between the Fourier series of $f(x)$ and the periodic extension of $f(x)$. However, the Theorem is valid only for functions that are **piecewise smooth** (a stricter condition than pwc).

Definition: Piecewise smooth (pws). If $f(x)$ and $\frac{df}{dx}$ are piecewise continuous (pwc) on some partition of $[a,b]$ then $f(x)$ is piecewise smooth on $[a,b]$.

Fourier's Theorem. Let $g(x)$ be piecewise smooth on the interval $[-L, L]$ and periodic, with period $2L$. The Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

associated with $g(x)$ converges to

$$\frac{1}{2}\left(g(x^+) + g(x^-)\right)$$

at every $x \in (-\infty, \infty)$.

Notice that the theorem can only be applied to pws periodic functions. If we are given a pws function $f(x)$ on a fixed interval $[-L, L]$ then $\tilde{f}(x)$ is both pws and periodic. The periodic extension $\tilde{f}(x)$ agrees with $f(x)$ on $[-L, L)$ (the interval we care about). So, to learn about the Fourier series associated with $f(x)$, we can apply the theorem with $g(x) = \tilde{f}(x)$ (treat $f(x)$ as if it were a periodic function). Now, if $\tilde{f}(x)$ is a function that is continuous at x then

$$\frac{1}{2}\left(\tilde{f}(x^+) + \tilde{f}(x^-)\right) = \tilde{f}(x),$$

but at points where $\tilde{f}(x)$ jumps, the Fourier series converges to the **average** of $\tilde{f}(x^+)$ and $\tilde{f}(x^-)$. This explains why the Fourier series investigated on the handout converged to $1/2$ at $x \in \{-1, 1/2, 1\}$.

The Theorem is useful because we can use it to sketch a Fourier series of a given pws function $f(x)$ on $[-L, L]$ without having to compute the Fourier coefficients:

1. First, sketch the periodic extension of $f(x)$.
2. At points of discontinuity, mark the average value,
3. Pull out the piece of the graph that corresponds to the interval $[-L, L]$ of interest.

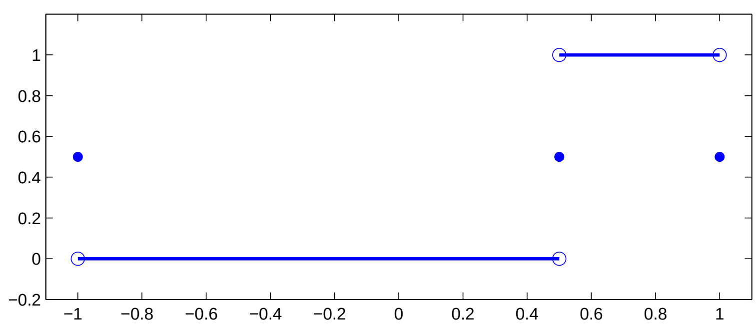


Figure 3: Fourier series of the function $f(x)$ on the interval $[-1, 1]$.

Example. Sketch the Fourier series associated with the function

$$f(x) = \begin{cases} 0 & -1 \le x \le 1/2 \\ 1 & 1/2 < x \le 1 \end{cases}.$$

Following the above steps, we obtain the picture shown in Figure 3.

2.4 Fourier Sine and Cosine Series

Fourier series of odd and even function have special forms.

Definition: Odd function. A function $f(x)$ is odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Examples: x, x^{17}, $\sin(x)$ are odd functions.

Definition: Even function. A function $f(x)$ is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$.

Examples: x^2, x^{100}, $\cos(x)$ are even functions.

Consider the integral of an odd or an even function over a symmetric interval of the form $[-L, L]$. In general,

$$f(x)\,\text{odd} \Rightarrow \int_{-L}^{L} f(x)\,dx = 0$$

and

$$f(x)\,\text{even} \Rightarrow \int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx.$$

Now, suppose $g(x)$ is odd. The associated Fourier coefficients are:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} g(x)\,dx = 0 \\
a_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{g(x)}_{odd}\underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{even}\,dx = 0 \\
b_n &= \frac{1}{L}\int_{-L}^{L} \underbrace{g(x)}_{odd}\underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{odd}\,dx = \frac{2}{L}\int_{0}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx.
\end{aligned}$$

Above, we have used the fact that the product of an odd and an even function is odd, while the product of two odd functions is even (can you prove this?). Since $a_0 = a_n = 0$, the cosine terms and the constant term in the Fourier series drop out, leaving only the sine terms. The Fourier series of an odd function is a sine series. Using the above calculation for b_n, the Fourier series has the form

$$\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_{0}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)\,dx\right)\sin\left(\frac{n\pi x}{L}\right).$$

Similarly, the Fourier series of an even function is called a cosine series, since all the coefficients in front of the sine terms are zero (see Exercise Sheet 3).

Now, consider the following question: If a function $f(x)$ is **not** odd, can we still represent it as a Fourier sine series? We will need to be able to do this when we apply the method of Separation of Variables, later on. Suppose $f(x)$ is pws on the half interval $[0, L]$. We can extend $f(x)$ to an odd function on the interval $[-L, L]$ (create an 'odd version' of it) in such a way that the extended function agrees with $f(x)$ on the original interval $[0, L]$. The Fourier series associated with this extended odd function will be a sine series. Similarly, any function $f(x)$ defined on $[0, L]$ can be extended to an even function on $[-L, L]$ and the associated Fourier series will be a cosine series.

Definition: Odd extension. Given $f(x)$ on $[0, L]$, the odd extension is

$$f_{\text{odd}}(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}.$$

Definition: Even extension. Given $f(x)$ on $[0, L]$, the even extension is

$$f_{\text{even}}(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}.$$

Examples. Figure 4 shows two functions defined on $[0, 1]$ and their odd and even extensions on $[-1, 1]$.

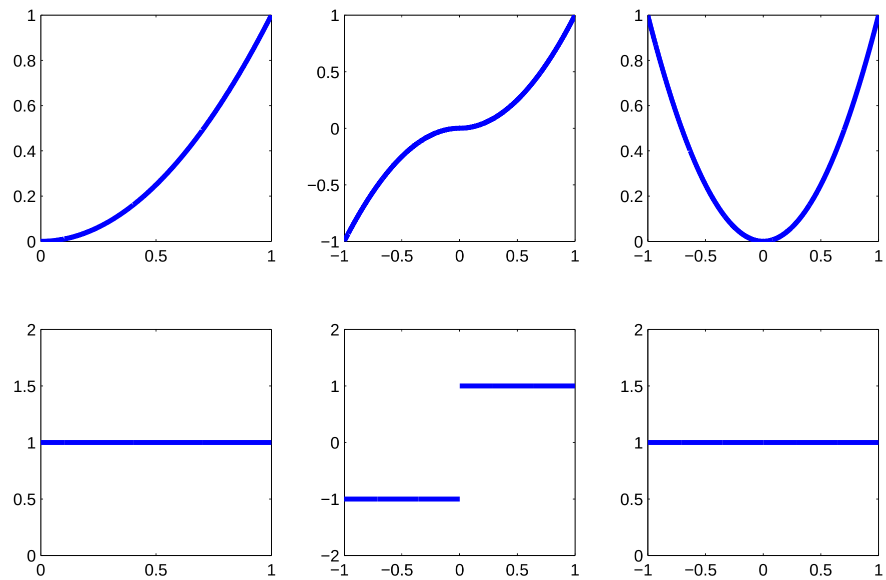


Figure 4: Left: $f(x)$ defined for $x \in [0, 1]$. Middle: odd extension $f_{\text{odd}}(x)$ on $[-1, 1]$. Right: even extension $f_{\text{even}}(x)$ on $[-1, 1]$.

Once again, using Fourier's Theorem, we can sketch a Fourier sine or cosine series, without integrating to find the coefficients.

Example. Find the Fourier sine series associated with the function $f(x) = 1$ on $[0, 1]$.

First, we sketch the Fourier sine series by applying the following steps.

1. Sketch $f(x)$ on $[0, 1]$ (see the bottom left plot in Figure 4).
2. Sketch $f_{\text{odd}}(x)$ on $[-1, 1]$ (see the bottom middle plot in Figure 4).
3. Sketch the periodic extension $\tilde{f}_{\text{odd}}(x)$ on $(-\infty, \infty)$.
4. At points of discontinuity, mark the average value (i.e., apply Fourier's Theorem).
5. Extract the piece of the graph that corresponds to the interval $[0, 1]$ of interest.

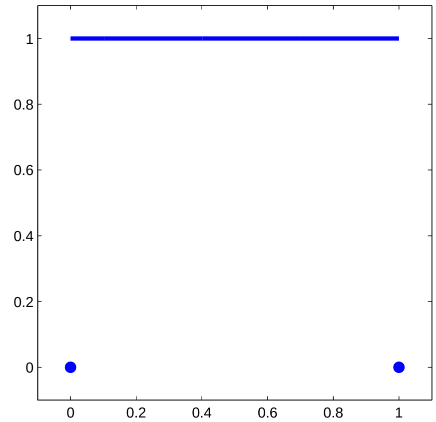


Figure 5: Fourier sine series associated with the function $f(x) = 1$ on $[0, 1]$.

This gives the graph in Figure 5. Notice that the Fourier sine series agrees with $f(x)$ in the interval $(0, 1)$ but converges to zero at the end points. To work out the sine series explicitly, we need to compute the coefficients of the Fourier series associated with the odd extension

$$f_{\text{odd}}(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 & -1 \leq x < 0 \end{cases}.$$

Since this is an odd function, we know $a_0 = 0 = a_n$ and

$$b_n = \frac{2}{1} \int_0^1 f_{\text{odd}}(x) \sin(n\pi x)\; dx = 2 \int_0^1 \sin(n\pi x)\, dx = \frac{2}{n\pi} \left(1 - (-1)^n\right).$$

Hence, the Fourier sine series is

$$\sum_{n=1}^{\infty} \frac{2}{n\pi} \left(1 - (-1)^n\right) \sin(n\pi x).$$

You should compare this with the Fourier cosine series of the same function (see Exercise Sheet 3).