MATH20411 PDEs and Vector Calculus B

Dr Stefan Güttel

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based on notes provided by Dr Catherine Powell.

To prepare for the more challenging topics that we will meet later in the course, we need to start off with some introductory material. In this section we will introduce two new co-ordinate systems (cylindrical and spherical co-ordinates), and briefly review the meaning of partial differentiation and the chain rule for partial derivatives. By the end of this section, you should be able to convert Cartesian co-ordinates into cylindrical and spherical co-ordinates and convert partial differential equations presented in Cartesian co-ordinates to partial differential equations in alternative co-ordinate systems. For further reading and examples, take a look at the book Calculus by James Stewart (a recommended text for your first year calculus courses). **It is recommended that you also review any notes you have from first year on polar co-ordinates and partial differentiation**.

1.1. Co-ordinate Systems

In three space dimensions, a point P may be represented in the Cartesian system by the co-ordinates (x, y, z) or as a vector of the form $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

are the standard basis vectors in $\mathbb{R}^3$.

Cartesian co-ordinates are convenient to describe objects such as rectangles and bricks, or objects composed of different sized bricks. For example, we understand that the set of points

$$\{(x, y, z) \,|\, 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$$

describes the unit cube (a cube with sides of length one, with vertices at $(0, 0, 0)$, $(0, 1, 0)$, etc.). To describe objects and surfaces with curvature, such as circles, spheres, cylinders, cones etc., it is often simpler to work in alternative co-ordinate systems.

Revision: polar co-ordinates

You already know that polar co-ordinates (r, θ) are useful for describing flat, circular geometries in **two** space dimensions. The Cartesian co-ordinates (x, y) of a point P tell us the distances to travel from the origin along the x and y axes to locate P. The co-ordinates (r, θ) give us the length of the line that connects P to the origin, and the angle between that line and the positive x-axis (measured in an anti-clockwise direction from the positive x-axis). Note that $r \ge 0$ and $0 \le \theta < 2\pi$. Using basic trigonometry (see Figure 1), if we know the polar co-ordinates (r, θ), then the Cartesian co-ordinates are given by

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Similarly, if we know the Cartesian co-ordinates, then

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}(y/x).$$

In three dimensions, there are two other useful co-ordinate systems: cylindrical and spherical co-ordinates.

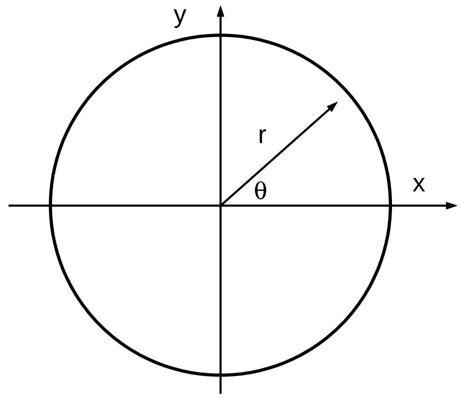

Figure 1: In $\mathbb{R}^2$, a point can be represented by Cartesian co-ordinates (x, y) or polar co-ordinates (r, θ).

Cylindrical co-ordinates

These are a natural extension of polar co-ordinates to three dimensions. Any point P in $\mathbb{R}^3$ can be represented by

$$(r, \theta, z)$$

where[1] (r, θ) are the standard polar co-ordinates of the point that is the projection of P onto the x-y plane, and z is the height (above or below the x-y plane) of P. Viewed another way, this says that any point P in three dimensions can be thought of as lying on the surface of a cylinder of some radius r, of infinite height, with the z-axis running through the centre. The co-ordinates θ and z tell us where on the surface of the cylinder the point is located.

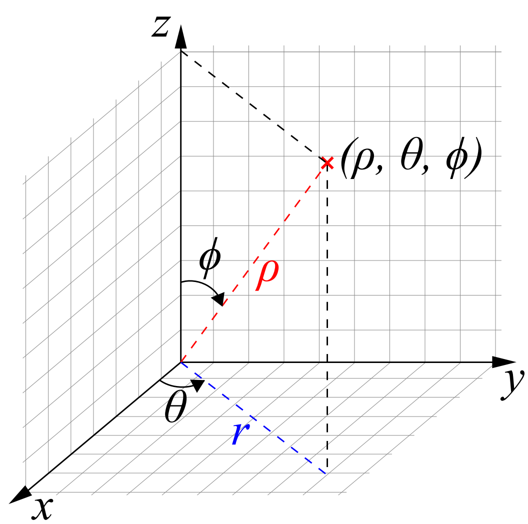

Figure 2: In $\mathbb{R}^3$, a point can be represented by cylindrical co-ordinates (r, θ, z) or spherical co-ordinates (ρ, θ, ϕ).

[1]Note that some textbooks use different letters in place of r and θ for cylindrical co-ordinates. We will use the notation that is consistent with what we already know for polar co-ordinates.

With the aid of simple trigonometry (see Figure 2), if we know the cylindrical co-ordinates of a point P then the Cartesian co-ordinates are given by:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$$

Similarly, if we know the Cartesian co-ordinates, the cylindrical co-ordinates are given by

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}(y/x), \qquad z = z.$$

The co-ordinate z is the same in both systems. As with polar co-ordinates, we must be careful when applying $\tan^{-1}$. Recall that tan is a periodic function, so there are infinitely many values θ such that $\tan(\theta)$ is equal to a given value of y/x. The value of θ we want must lie in the range $[0, 2\pi)$.

Example. Find the cylindrical co-ordinates of the point $(x, y, z) = (3, -3, -1)$.

Answer: Clearly, $z = -1$ and $r = \sqrt{9+9} = \sqrt{18}$. Direct calculation (using a calculator) gives:

$$\theta = \tan^{-1}(-3/3) = \tan^{-1}(-1) = -\pi/4.$$

This is outside the desired range. In fact, we have

$$\tan^{-1}(-1) = -\pi/4 + n\pi,$$

for any $n \in \mathbb{N}$. Here, $\theta = -\pi/4 + 2\pi = 7\pi/4$ (or 315 degrees). Sketch a diagram to convince yourself that this is the correct value! There are also plenty more examples to try on Exercise Sheet 1.

Spherical co-ordinates

These co-ordinates are useful for geometries with radial symmetry (e.g., spheres). Any point P can be thought of as lying on the surface of a sphere, of some radius ρ, centred at the origin (see Figure 2). If (x, y, z) are the Cartesian co-ordinates of P then the distance from P to the origin is

$$\rho = \sqrt{x^2 + y^2 + z^2}.$$

This is the radius of the sphere upon whose surface P lies. We need to supply two other co-ordinates (angles) to uniquely specify the location of P. The spherical co-ordinates are (ρ, θ, ϕ), where

- ρ is the distance from P to the origin.
- θ is the same angle used in the polar and cylindrical systems. That is, if we project P onto the (x, y) plane, θ is the angle measured anti-clockwise from the positive x-axis to the line joining the projected point to the origin.
- ϕ is the angle between the positive z-axis and the line connecting P to the origin.

Note that $\rho > 0$ (if $\rho = 0$ then we are at the origin), $0 \le \theta < 2\pi$, and $0 \le \phi \le \pi$. You might like to think about the angles ϕ and θ as longitude and latitude.

With some basic trigonometry, we find

$$r = \rho\sin\phi$$

and so

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta.$$

In addition

$$z = \rho \cos \phi.$$

Alternatively, if we know the Cartesian co-ordinates (x, y, z) then,

$$\rho = \sqrt{x^2 + y^2 + z^2}, \qquad \theta = \tan^{-1}(y/x).$$

Clearly $\tan \phi = r/z$ and so the angle ϕ can be calculated by

$$\phi = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right).$$

Example. Find the spherical co-ordinates of the point $(x, y, z) = (3, -3, -1)$.

Answer: We have $\rho = \sqrt{9 + 9 + 1} = \sqrt{19}$ and we already know $\theta = 7\pi/4$ (or 315 degrees, from the previous example). Finally, $\phi = \tan^{-1}\left(\sqrt{18}/-1\right) = -1.3393 + n\pi$. The correct value is achieved with $n = 1$, giving $\phi = 1.8023$ radians (to 4 d.p.) or $\phi = 103.26$ degrees (to two d.p). Now try the other examples on Exercise Sheet 1!

You should now be able to describe standard surfaces like planes, cones, spheres etc in Cartesian, cylindrical and/or spherical co-ordinates. Sometimes, working in a particular co-ordinate system vastly simplifies the equation for the surface.

Example. What surface is represented by the equation $\rho = 2$? How would the same surface be represented in Cartesian co-ordinates?

Answer: $\rho = 2$ is the equation for a sphere of radius 2, centred at the origin. More precisely, $\rho = 2$ means the set of points (ρ, θ, ϕ) for which $\rho = 2$ is fixed and θ and ϕ take any of their permitted values. In Cartesian co-ordinates, the same surface is represented by the set of points (x, y, z) such that

$$\sqrt{x^2 + y^2 + z^2} = 2.$$

As you can see, the equation is far simpler in spherical co-ordinates.

1.2. Partial Differentiation

A 'differential equation' is an equation containing an unknown function, say u, and some of its derivatives. If u is a function of a single variable x, then the derivatives are

$$\frac{du}{dx}, \quad \frac{d^2u}{dx^2}, \quad \frac{d^3u}{dx^3}, \quad \ldots.$$

A differential equation associated with a function of **single** variable is called an **ordinary differential equation** or **ODE**. For example[2]

$$\frac{d^2u}{dx^2} + \lambda u = 0.$$

[2]You studied how to solve second-order ODEs with constant coefficients in the first year and it will be very important in this course that you remember how to do this. If you have forgotten, please revise it now.

The first derivative $\frac{du}{dx}$ represents the rate of change of u with respect to x. A more formal definition is: the slope of the tangent line to the curve $y = u(x)$ at a given point x. For example, consider the simple function $u(x) = x^2$. We all know the derivative is $2x$. Using the formal definition,

$$\frac{du}{dx} = \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} 2x + h = 2x.$$

If u is a function of **more than one** variable, e.g., $u(x, y)$, then there are other derivatives to consider. We can calculate the rate of change of u with respect to either x or y. The first **partial derivatives** are denoted

$$\frac{\partial u}{\partial x}, \qquad \frac{\partial u}{\partial y},$$

or u_x, u_y, for short. There are three second partial derivatives, including

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right),$$

which is usually abbreviated to u_{xy}, etc. Partial derivatives are rates of change with respect to one variable, whilst the other variables are held fixed.

Example. Consider the function

$$u(x, y) = \left(x^2 + 3y^2\right) \exp\left(-x^2 - y^2\right).$$

To find the partial derivative u_x, recall that we treat y like a constant and differentiate as if u were only a function of x as follows

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \left(x^2 + 3y^2\right)(-2x) \exp\left(-x^2 - y^2\right) + \exp\left(-x^2 - y^2\right)(2x) \\
&= \left(2x - 2x^3 - 6xy^2\right) \exp\left(-x^2 - y^2\right) \\
&= 2x\left(1 - x^2 - 3y^2\right) \exp\left(-x^2 - y^2\right).
\end{aligned}$$

This is the rate of change of u with respect to x if y is kept fixed. For example, if $y = 0$, then

$$\frac{\partial u}{\partial x}\,|_{y=0} = 2x\left(1 - x^2\right) \exp\left(-x^2\right).$$

Similarly, $\frac{\partial u}{\partial y}$ is found by treating x as a constant, and differentiating with respect to y.

Note that if $u = u(x, y)$ is a function of two variables then fixing a value for y amounts to taking a cross-section through the surface $z = u(x, y)$. The partial derivative u_x actually gives the slope of the tangent to the curve that is that cross-section. We can plot the function $u(x, y)$ from the above example easily in MATLAB, by fixing values for x and y and then interpreting $z = u(x, y)$ as the output of the function. Try the following MATLAB commands.

```
>> x=linspace(-4,4,64); y=linspace(-4,4,64); [xx,yy]=meshgrid(x,y);
>> zz=(xx.^2+3.*yy.^2).*exp(-xx.^2-yy.^2);
>> surf(xx,yy,zz); axis square; shg
```

If we set $y = 0$ then $u(x, 0) = x^2 \exp(-x^2)$ is a cross-section through the surface $z = u(x, y)$ (this is shown in the right plot in Figure 3). In the example above, we found $\frac{\partial u}{\partial x}\,|_{y=0}$, which is the slope of the tangent line to this cross-section.

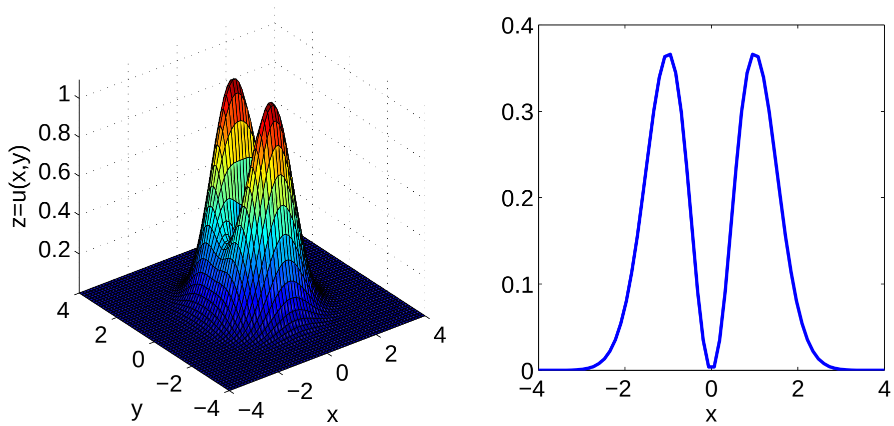

Figure 3: Left: MATLAB `surf` plot of the function $u(x,y) = \left(x^2 + 3y^2\right)\exp\left(-x^2 - y^2\right)$. Right: cross-section through $z = u(x,y)$ at $y = 0$.

Formally, the first partial derivatives of a function u of two variables x and y are given by

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(x+h,y) - u(x,y)}{h}, \qquad \frac{\partial u}{\partial y} = \lim_{h\to 0}\frac{u(x,y+h) - u(x,y)}{h}.$$

Example. Use the above definition to find the first partial derivatives of the function $u(x,y) = x^2 + y^2$.

Answer: Only the calculation for u_x is shown here. u_y is left as an exercise. We have

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{(x+h)^2 + y^2 - \left(x^2 + y^2\right)}{h} = \lim_{h\to 0}\frac{2xh + h^2}{h} = 2x.$$

1.3. The Chain Rule for Partial Derivatives

For functions of a single variable, we have the well-known chain rule. Suppose u is a function of t and t is a function of x. Then u is a function of x (because it depends on t) and

$$\frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx}.$$

Example. Suppose $u = \sin^2(t)$ and $x = \sin t$. Use the chain rule to find $\frac{du}{dx}$.

Answer: Differentiating u with respect to t and x with respect to t (which is natural since u is given as a function of t and x is given as a function of t) gives

$$\frac{du}{dt} = 2\sin t\cos t, \qquad \frac{dx}{dt} = \cos t.$$

Now, since[3] $\frac{dt}{dx} = \left(\frac{dx}{dt}\right)^{-1}$, the chain rule gives

$$\frac{du}{dx} = 2\sin t = 2x.$$

[3] for functions of **one** variable only!

(Of course, we could also have made the substitution $x = \sin t$ immediately to give $u = x^2$ and then differentiated with respect to x straightforwardly.)

A similar rule holds for functions of more than one variable. Here, we'll focus on functions of two variables. Suppose $u = u(s,t)$ and $s = s(x,y)$, and $t = (x,y)$. Then, u is a function of x and y through its dependence on s and t. The chain rule gives:

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial t}\frac{\partial t}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x},\\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial t}\frac{\partial t}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y}.\end{aligned}$$

Notice that there are two terms in each expression. This is because the function u depends on x and y in two distinct ways: via s and via t.

Example. Suppose $u = t\sin s$ and $s = x^2 + y^2$, $t = 2x + 4y$. Use the chain rule to find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

Answer: Applying the chain rule gives

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2\sin s + 2xt\cos s = 2\sin\left(x^2+y^2\right) + 2x\left(2x+4y\right)\cos\left(x^2+y^2\right),\\ \frac{\partial u}{\partial y} &= 4\sin s + 2yt\cos s = 4\sin\left(x^2+y^2\right) + 2y\left(2x+4y\right)\cos\left(x^2+y^2\right).\end{aligned}$$

Note that for functions of more than one variable, it is very important to realise that in general $\frac{\partial x}{\partial t} \neq \left(\frac{\partial t}{\partial x}\right)^{-1}$! For instance, suppose x and t are both functions of two variables, say x depends on t and s, and t depends on x and y. Then, $\frac{\partial x}{\partial t}$ is the rate of change of x with respect to t when s is fixed but $\frac{\partial t}{\partial x}$ is the rate of change of t with respect to x when y is fixed.

We can use the chain rule to convert derivatives in one co-ordinate system to partial derivatives in another co-ordinate system.

Example. Suppose that a calculation or a given equation involves the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ where x and y are understood to be Cartesian co-ordinates. Now suppose that it is preferable to think of u as a function of polar co-ordinates. Convert the derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ into derivatives with respect to r and θ.

Answer: We know that r and θ are both functions of x and y. That is,

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}\left(y/x\right).$$

When viewed as a function of polar co-ordinates, u depends on x via its dependence on r and θ. Hence, the chain rule gives:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}.$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

Differentiating the expressions for r and θ with respect to x and y gives (**exercise**)

$$\frac{\partial u}{\partial x} = \cos\theta\frac{\partial u}{\partial r} - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta},$$

and similarly,

$$\frac{\partial u}{\partial y} = \sin\theta\frac{\partial u}{\partial r} + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta}.$$

Laplace's equation is a classical differential equation that we will study later in the course. [**See the handout on classical PDEs.**] In two dimensions, this is written as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

To convert this equation into say, polar co-ordinates, we need to convert the second derivatives with respect to x and y into derivatives with respect to r and θ. Given the above expressions for the first derivatives, you can now attempt this (see the last question on Exercise Sheet 1).

1.4. Solutions to PDEs

Solutions to ODEs usually have arbitrary constants in them. For example, consider the first-order ODE,

$$\frac{du}{dx} = -\lambda x.$$

To solve this, we can integrate both sides with respect to x to obtain

$$u(x) = \int -\lambda x\, dx = -\frac{\lambda x^2}{2} + C,$$

where C is any constant (note that this is an indefinite integral). In addition, consider

$$\frac{dv}{dx} = -\lambda v.$$

The solution is $v(x) = Ce^{-\lambda x}$, where C is any constant. In both cases, **infinitely** many solutions exist. To obtain a **unique** solution, extra conditions are needed. For example, if we know that $u(0) = 1$ in the first example, then

$$-\lambda(0) + C = 1 \Rightarrow C = 1,$$

and so the unique solution is

$$u(x) = -\frac{\lambda x^2}{2} + 1.$$

Solutions to **partial differential equations (PDEs)** contain arbitrary functions in place of arbitrary constants. For example, consider the simple PDE

$$\frac{\partial u}{\partial x} = 0$$

where we assume that u is a function of x and y. Integrating both sides with respect to x gives

$$u(x, y) = \int 0\, dx = C(y),$$

where $C(y)$ is an arbitrary function of y (this acts like a constant when differentiating with respect to x). Check - any function of y (only) that you can think of satisfies the above partial differential equation. To obtain a unique solution, we need extra conditions such as $u(x, 0) = f(x)$ or $u(x, 1) = g(x)$ where $f(x), g(x)$ are specified functions.

Example. Find the unique solution $u(x, t)$ to the PDE $\frac{\partial u}{\partial x} = t + 1$ such that $u(0, t) = f(t)$.

Answer: Integrating with respect to x gives

$$u(x, t) = xt + x + C(t),$$

where $C(t)$ is an arbitrary function of t. Since $u(0, t) = f(t)$, we know $C(t) = f(t)$ (a specific function) so the unique solution is

$$u(x, t) = x(t + 1) + f(t).$$