1. Second-order PDEs

Recall that a linear second-order PDE for a function of two variables $u(x, t)$ (where x denotes the spatial co-ordinate and t is time) can be written in the form

$$Lu(x,t) = Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu \;=\; G,$$

for given values of x in some interval $[p, q] \subset \mathbb{R}$ and for values $t \geq 0$, and where the coefficients A–F are functions of t and x only.

L is a **linear** PDE operator, if, for any two functions $u(x, t)$ and $v(x, t)$,

$$L(c_1u(x,t) + c_2v(x,t)) = c_1Lu(x,t) + c_2Lv(x,t),$$

where c_1 and c_2 are arbitrary constants.

The PDE is solved subject to some **initial conditions** (if t represents time) such as,

$$u(0,x) = u_0(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(0,x) = u_1(x),$$

and some **boundary conditions** such as,

$$\alpha_1 u(t,p) + \beta_1 \frac{\partial u}{\partial x}(t,p) = h(t),$$
$$\alpha_2 u(t,q) + \beta_2 \frac{\partial u}{\partial x}(t,q) = k(t),$$

where at least one of α_1 and β_1 must be non-zero and at least one of α_2 and β_2 must be non-zero.

- When is $Lu = g$ a **homogeneous** PDE?
- When are the boundary conditions in homogeneous form?

To investigate the method of **separation of variables**, we ignore (temporarily) the initial conditions and consider **linear, homogeneous second-order PDEs with linear homogeneous boundary conditions at $x = p$ and $x = q$**. That is, we focus on,

$$Lu(x,t) = Au_{xx} + Bu_{xt} + Cu_{tt} + Du_x + Eu_t + Fu \;=\; 0,$$

subject to,

$$\alpha_1 u(t,p) + \beta_1 \frac{\partial u}{\partial x}(t,p) = 0,$$
$$\alpha_2 u(t,q) + \beta_2 \frac{\partial u}{\partial x}(t,q) = 0.$$

It is important to recognise the conditions under which the method of separation of variables can be applied. We cannot use it to solve ALL PDEs!

2. Key Points

- Separation of variables is a method for finding *all* possible solutions of the linear homogeneous problem, $Lu = 0$, only, subject to linear homogeneous boundary conditions, e.g.

$$\alpha_1 u(t,p) + \beta_1 \frac{\partial u}{\partial x}(t,p) = 0, \quad \alpha_2 u(t,q) + \beta_2 \frac{\partial u}{\partial x}(t,q) = 0.$$

- The trivial function $u(x,t) \equiv 0$ is *always* a solution of a linear, homogeneous PDE.
- If $u(x,t)$ and $v(x,t)$ are *any* solutions of the linear, homogeneous problem, then so is *any* linear combination, $c_1 u(x,t) + c_2 v(x,t)$, of them. (Can you prove this?)
- The method of separation of variables assumes that solutions take the particular form,

$$u_n(x,t) = X_n(x) T_n(t), \quad n = 1, 2, 3, \ldots,$$

  where $X_n(x)$ is a function of x only and $T_n(t)$ is a function of t only.
- Using the method, general solutions are written as an *infinite* linear sum of the form,

$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t).$$

- The coefficients in the sum are recovered by imposing the initial conditions and using ideas from Fourier Series.

3. The Method

We will spend a lot of time in lectures going through all the steps of the method of Separation of Variables for the Heat Equation, the Wave Equation, and Laplace's Equation. The basics of the general method are summarised below and will help you to revise at the end of the course.

First, let us consider a *single* solution, $u(x,t) = X(x)T(t)$ of the linear, homogeneous problem. We already know that $u(x,t) \equiv 0$ is a solution, so let us assume that $X(x) \not\equiv 0$ and $T(t) \not\equiv 0$ so that we obtain something more interesting.

Boundary conditions. Substituting $u(x,t) = X(x)T(t)$ into the boundary conditions gives,

$$\alpha_1 X(p)T(t) + \beta_1 T(t) \frac{dX}{dx}(p) = 0,$$
$$\alpha_2 X(q)T(t) + \beta_2 T(t) \frac{dX}{dx}(q) = 0.$$

Since $T(t) \not\equiv 0$, we can divide by $T(t)$ to obtain boundary conditions for $X(x)$ only, i.e.

$$\alpha_1 X(p) + \beta_1 \frac{dX}{dx}(p) = 0,$$
$$\alpha_2 X(q) + \beta_2 \frac{dX}{dx}(q) = 0.$$

PDE. Substituting $u(x,t) = X(x)T(t)$ into the PDE $Lu = 0$ gives,

$$AT \frac{d^2 X}{dx} + B \frac{dT}{dt} \frac{dX}{dx} + CX \frac{d^2 T}{dt^2} + DX \frac{dT}{dt} + ET \frac{dX}{dx} + FTX = 0,$$

which is an ODE! Again, since $T(t) \not\equiv 0$ and $X(x) \not\equiv 0$, we can divide by XT to obtain,

$$A \frac{1}{X} \frac{d^2 X}{dx} + B \frac{1}{TX} \frac{dT}{dt} \frac{dX}{dx} + C \frac{1}{T} \frac{d^2 T}{dt^2} + D \frac{1}{T} \frac{dT}{dt} + E \frac{1}{X} \frac{dX}{dx} + F = 0.$$

Now, the method of separation of variables can only be applied if certain conditions are met. In essence, each of the coefficient terms must be a function of only t or only x.

We require:

$$F(x,t) = F_1(x) + F_2(t), E = E(x), D = D(t), C = C(t), B \equiv 0, A = A(x).$$

In other words,

- the PDE must have no mixed derivatives
- coefficients for the terms with x-derivatives must only be functions of x
- coefficients for the terms with t-derivatives must only be functions of t
- the coefficient for the term with no derivatives can be the sum of functions of t and x

Note that at first glance, a given PDE may not satisfy all these conditions. However, it may be possible to rearrange it into the appropriate form.

Exercise: Which of the following PDEs might be solvable using the method of separation of variables?

- $\frac{\partial^2 u}{\partial t^2} + (t^2 + x^2)\frac{\partial^2 u}{\partial x^2} = 0$
- $t^2 \frac{\partial^2 u}{\partial t^2} + x^2 \frac{\partial^2 u}{\partial x^2} = 0$
- $x^2 \frac{\partial^2 u}{\partial t^2} + t^2 \frac{\partial^2 u}{\partial x^2} = 0$
- $\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} - (1 - x^2)u = 0$
- $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + (t - x)u = 0$
- $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial t \partial x} + \frac{\partial^2 u}{\partial x^2} - u = 0$

If all the conditions are met, we obtain,

$$F_1(x) + F_2(t) + D(t)\frac{1}{T}\frac{dT}{dt} + E(x)\frac{1}{X}\frac{dX}{dx} + C(t)\frac{1}{T}\frac{d^2T}{dt^2} + A(x)\frac{1}{X}\frac{d^2X}{dx} = 0.$$

The key point is that we can now *separate* the dependence on x and t to opposite sides of the equation and write them each equal to a *separation constant*, λ, such that,

$$F_1(x) + \frac{E(x)}{X}\frac{dX}{dx} + \frac{A(x)}{X}\frac{d^2X}{dx} = -F_2(t) - \frac{D(t)}{T}\frac{dT}{dt} - \frac{C(t)}{T}\frac{d^2T}{dt^2} = \lambda$$

- Why is λ a constant?

Hence, we obtain 2 ODEs to solve, one for $X(x)$ and one for $T(t)$. Recall that $X(x)$ must also satisfy the boundary conditions. The problem to be solved for $X(x)$ is, then:

$$A(x)\frac{d^2X}{dx} + E(x)\frac{dX}{dx} + F_1(x)X + \lambda X = 0, \qquad (1)$$

subject to the boundary conditions,

$$\alpha_1 X(p) + \beta_1 \frac{dX}{dx}(p) = 0,$$
$$\alpha_2 X(q) + \beta_2 \frac{dX}{dx}(q) = 0.$$

Similarly, the problem to be solved for $T(t)$ is,

$$C(t)\frac{d^2T}{dt^2} + D(t)\frac{dT}{dt} + F_2(t)T - \lambda T = 0. \tag{2}$$

Recall that in the course MATH10232 (Calculus and Applications), you studied ordinary differential equations (ODEs) and methods for solving equations like (1) and (2) exactly. **It will be a good idea to read over those lecture notes.**

4. General theory

The values of λ for which non-trivial solutions to the ODEs (1) and (2) exist, are called **eigenvalues**, while the corresponding solutions $X(x)$ are called **eigenfunctions**. Many general results about the eigenvalues and eigenfunctions can be proved. For instance,

- there are infinitely many eigenvalues $\lambda_1, \lambda_2, \ldots,$
- all the eigenvalues are **real**,
- for each eigenvalue λ_n, there is at least one corresponding eigenfunction $X_n(x)$ and at least one function $T_n(t)$,
- given the eigenvalues and eigenfunctions, the general solution is of the form

  $$u(x,t) = \sum_{n=1}^{\infty} c_n T_n(t) X_n(x),$$

- the different eigenfunctions are **orthogonal** with respect to an inner-product of the form

  $$(w\, X_n, X_m) = \int_p^q w(x) X_n(x) X_m(x)\, dx = 0 \quad (\text{for } \; n \neq m)$$

  where $w(x)$ is a *weight function*,
- the orthogonality property can be used to determine the coefficients c_n in the series solution for $u(x,t)$ that are needed to satisfy the initial conditions (that we ignored at the beginning).

We will look at some of these properties in the lectures.