To develop the idea of a Fourier series, we will need to study functions $f(x)$, $g(x)$ which are 'orthogonal.' In the linear algebra course, you met orthogonal vectors $\mathbf{f}$, $\mathbf{g} \in \mathbb{R}^n$. Before extending this idea to functions, it is useful to review the key definitions and ideas for vectors.

Definition (inner-product on $\mathbb{R}^n$) The inner-product (also scalar product or dot product) of two vectors $\mathbf{f}, \mathbf{g} \in \mathbb{R}^n$, written $\langle \mathbf{f}, \mathbf{g} \rangle$ or $\mathbf{f} \cdot \mathbf{g}$ or $\mathbf{f}^T\mathbf{g}$ is defined via

$$\langle \mathbf{f}, \mathbf{g} \rangle = \sum_{i=1}^{n} f_i g_i,$$

where f_i, g_i denote the i^{th} components of $\mathbf{f}$ and $\mathbf{g}$.

Example Consider the vectors $\mathbf{f} = [1, 2, 10]^T$, $\mathbf{g} = [3, -2, 0]^T$ in $\mathbb{R}^3$. Their inner-product is $\langle \mathbf{f}, \mathbf{g} \rangle = (1 \times 3) + (2 \times -2) + (10 \times 0) = 3 - 4 + 0 = -1$.

If $\langle \mathbf{f}, \mathbf{g} \rangle = 0$ then the vectors $\mathbf{f}$ and $\mathbf{g}$ are perpendicular or **orthogonal**. Using the inner-product, we can also define a norm or measure on $\mathbb{R}^n$.

Definition (norm on $\mathbb{R}^n$) Let $\mathbf{f} \in \mathbb{R}^n$. The norm of $\mathbf{f}$ is

$$\| \mathbf{f} \| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle} = \left(\sum_{i=1}^{n} f_i^2 \right)^{\frac{1}{2}}.$$

Example The norm of the vector $\mathbf{f}$ in the above example is $\| \mathbf{f} \| = \sqrt{1^2 + 2^2 + 10^2} = \sqrt{105}$.

Note that for any pair of vectors $\mathbf{f}$, $\mathbf{g} \in \mathbb{R}^n$, $\| \mathbf{f} - \mathbf{g} \|$ is the Euclidean distance between $\mathbf{f}$ and $\mathbf{g}$. Now consider a set of n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ from $\mathbb{R}^n$.

Definition (orthogonal set) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal set of vectors in $\mathbb{R}^n$ if $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ whenever $i \neq j$ and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle \neq 0$ when $i = j$. If $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = \| \mathbf{v}_i \|^2 = 1$ for each $i = 1, 2, \ldots n$, then the vectors are normalised. If the vectors are orthogonal and normalised, they are said to be orthonormal.

Now suppose that $\mathbf{f}$ is *any* vector in $\mathbb{R}^n$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a given set of orthogonal basis vectors for $\mathbb{R}^n$. Then, (using the definition of basis) $\mathbf{f}$ can be written as a linear combination of those vectors.

$$\mathbf{f} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n, \qquad c_i \in \mathbb{R}, \quad i = 1, \ldots n.$$

To determine the coefficients, we can use the fact that the basis vectors are orthogonal. Taking the inner-product with the vector $\mathbf{v}_i$, (for any $i = 1, 2, \ldots, n$) on both sides of the equation gives

$$\langle \mathbf{f}, \mathbf{v}_i \rangle = c_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + c_2 \langle \mathbf{v}_2, \mathbf{v}_i \rangle + \ldots + c_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle.$$

Using orthogonality, we see that only one term on the right is non-zero and $\langle \mathbf{f}, \mathbf{v}_i \rangle = c_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle$. Solving for c_i (for each i) and substituting in the expansion of $\mathbf{f}$ gives

$$\mathbf{f} = \sum_{i=1}^{n} c_i \mathbf{v}_i = \sum_{i=1}^{n} \frac{\langle \mathbf{f}, \mathbf{v}_i \rangle}{\langle \mathbf{v}_i, \mathbf{v}_i \rangle} \mathbf{v}_i.$$

If the basis vectors are, additionally, normalised, so that $\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 1$, then we have

$$\mathbf{f} = \sum_{i=1}^{n} c_i \mathbf{v}_i = \sum_{i=1}^{n} \langle \mathbf{f}, \mathbf{v}_i \rangle \, \mathbf{v}_i.$$

The important message is that any vector $\mathbf{f}$ in $\mathbb{R}^n$ can be expressed as a linear combination of a set of orthogonal vectors, with coefficients that depend on $\mathbf{f}$ and defined using the inner-product. **Basically, orthogonal vectors are the building blocks from which all other vectors in $\mathbb{R}^n$ can be produced.**
