Consider the convection-diffusion equation:

$$-\frac{d^2u}{dx^2} + w\frac{du}{dx} = f(x), \quad 0 < x < 1 \quad \text{with} \quad u(0) = 1,\, u(1) = 0.$$

with $w > 0$ and $f = 0$. The exact solution is:

$$\frac{(\exp w - \exp(wx))}{(\exp w - 1)}.$$

In the plot below, exact solutions are shown for $w = 1, 5, 10, 20$. When w is large, the convection term dominates and the solution has a thin layer close to the right boundary point $x = 1$ where the solution drops from one to zero.

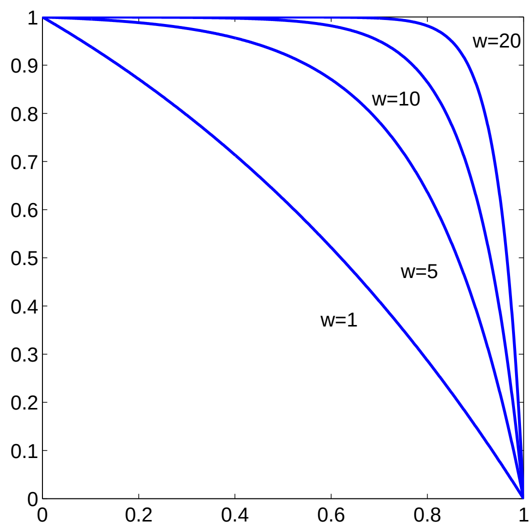

Figure 1: Exact solutions for $w = 1, 5, 10, 20$

Centered Finite Difference Scheme

Suppose we choose $N = 4$ intervals on $[0, 1]$ and set $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$ and $x_4 = 1$. Let U_j denote an approximation to the exact solution $u(x)$ at $x = x_j$, and denote $f_j = f(x_j)$ ($= 0$ in this example). Now, applying centered finite differences for both the first and second order derivatives in the original ODE leads to the scheme:

$$\left(-\frac{1}{h^2} - \frac{w}{2h}\right)U_{j-1} + \left(\frac{2}{h^2}\right)U_j + \left(-\frac{1}{h^2} + \frac{w}{2h}\right)U_{j+1} = f_j, \quad j = 1:3.$$

Since $h = \frac{1}{4}$ and **the boundary conditions give** $U_0 = 1$ **and** $U_4 = 0$, we obtain three equations for the unknown values U_1, U_2, U_3:

$$\begin{aligned} -16 - 2w\,(1) + 32U_1 + (-16 + 2w)\,U_2 &= 0 \\ -16 - 2wU_1 + 32U_2 + (-16 + 2w)\,U_3 &= 0 \\ -16 - 2wU_2 + 32U_3 + 0 &= 0 \end{aligned}$$

$$\Rightarrow \begin{pmatrix} 32 & -16 + 2w & 0 \\ -16 - 2w & 32 & -16 + 2w \\ 0 & -16 - 2w & 32 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 16 + 2w \\ 0 \\ 0 \end{pmatrix}.$$

Notice how the non-zero boundary condition for U_0 affects the right-hand side vector now.

For a given value of the 'wind' w we can solve this 3×3 linear system (using Gaussian Elimination by hand or in MATLAB using `trisolve.m`). For larger values of N there will be $N-1$ equations to solve and we will need the help of a computer to solve the system. Below, we've plotted the numerical approximations obtained for $w = 40$ and for four different numbers of intervals. In each case, we also plot the exact solution for comparison.

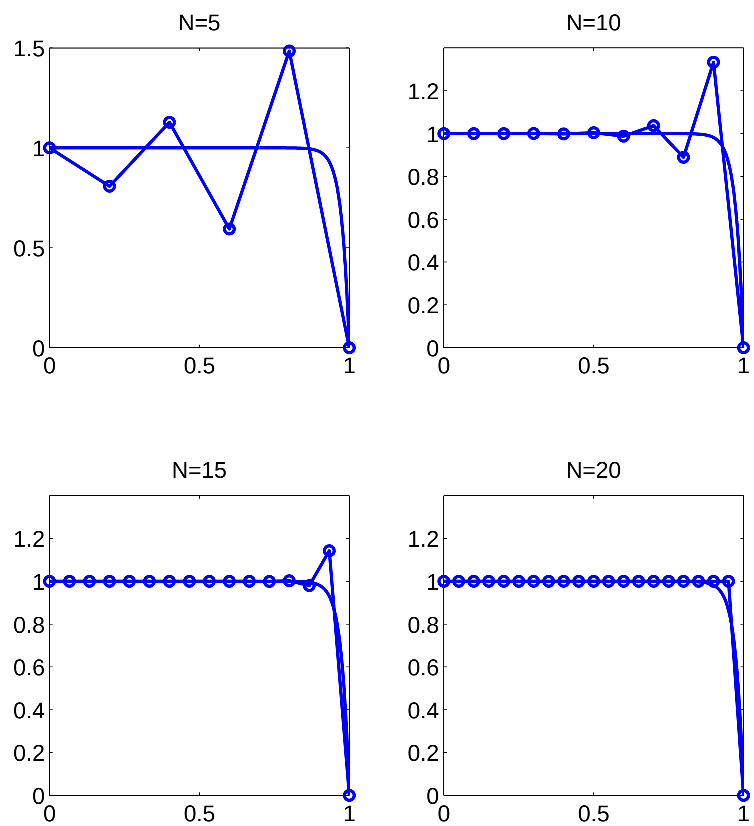

Figure 2: Exact solution and numerical approx. to $u(x)$ for $\mathbf{w} = \mathbf{40}$ with $N = 5, 10, 15$ and 20 intervals.

Observe that if N is too small, we don't resolve the layer in the exact solution and the numerical approximation has lots of oscillations in it. (Compare these observations with some theory discussed in lectures.)
To generate these results yourself, download the MATLAB codes `trisolve.m` and `conv_diff_1D.m` from the course webpage. Save these files to your P: drive. Start up MATLAB and change directory so that your current working directory is your P: drive. Open the file `conv_diff_1D.m` in the editor window - or else just type

```
>> type conv_diff_1D
```

at the MATLAB prompt. Read the file to understand what it does. To run the program, you type:

```
[u,u_exactx,error]=conv_diff_1D(w,N)
```

at the prompt in the command window, inserting the desired values for w, (where $r = w^2$) and N, the number of intervals for the finite difference calculation. For the above example with $w = 40$, typing,

```
[u,u_exactx,error]=conv_diff_1D(40,5)
[u,u_exactx,error]=conv_diff_1D(40,10)   etc.
```

will produce the results in Figure 2. Try it!

Upwind Finite Difference Scheme
Alternatively, we can replace the approximation to the first derivative with a backward finite difference and obtain, in the case $N = 4$, the set of equations:

$$\left(-\frac{1}{h^2} - \frac{w}{h}\right) U_{j-1} + \left(\frac{2}{h^2} + \frac{w}{h}\right) U_j - \left(\frac{1}{h^2}\right) U_{j+1} = f_j, \quad j = 1:3.$$

Since $h = \frac{1}{4}$ and the boundary conditions give $U_0 = 1$ and $U_4 = 0$, we obtain three equations for the unknown values U_1, U_2, U_3:

$$\begin{array}{rcl} -16 - 4w\,(1) + (32 + 4w)\,U_1 + -16U_2 & = & 0 \\ -16 - 4wU_1 + (32 + 4w)\,U_2 - 16U_3 & = & 0 \\ -16 - 4wU_2 + (32 + 4w)\,U_3 + 0 & = & 0 \end{array}$$

$$\Rightarrow \begin{pmatrix} 32 + 4w & -16 & 0 \\ -16 - 4w & 32 + 4w & -16 \\ 0 & -16 - 4w & 32 + 4w \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} 16 + 4w \\ 0 \\ 0 \end{pmatrix}.$$

Compare this system with the one obtained for the centered scheme (on page 1). In MATLAB, you can repeat the experiment performed above for the centered scheme and plot the solutions obtained for the case $w = 40$ with $N = 5, 10, 15, 20$ etc. intervals.