Recall that a vector-function (or 'field') in 3 space dimensions, is a rule which tells us how to associate a *vector* with each point (x, y, z) e.g. the velocity field of a fluid.

In Cartesian co-ordinates, a vector function is:

$$\mathbf{F} = \mathbf{F}(x, y, z) = F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}.$$

$\mathbf{F}$ specifies both the *magnitude* (or 'speed') of the field as well as its *direction* at each point (x, y, z). The magnitude is given by:

$$|\mathbf{F}| = \sqrt{F_x^2 + F_y^2 + F_z^2}.$$

Note that F_x, F_y and F_z are functions of x, y and z in general. For short-hand, we usually omit to write this dependence explicitly.

We sketch vector functions as a collection of arrows, one for each point (x, y, z). The length of each arrow signifies the magnitude of the field at that point.

Two-dimensional examples

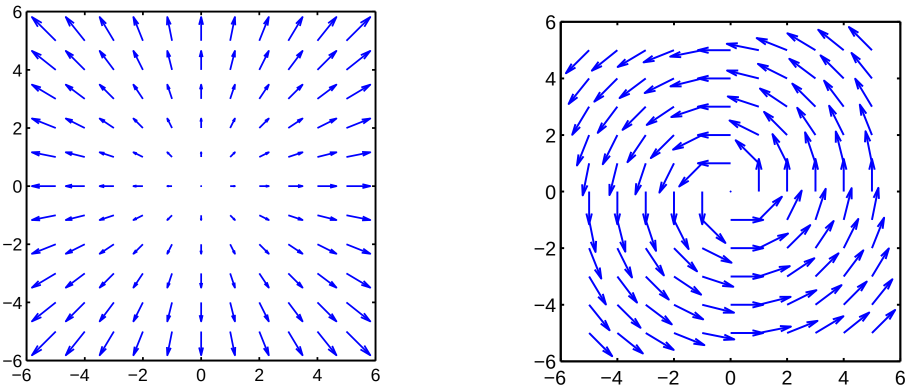

Figure 1: $\mathbf{F}_1 = x\mathbf{i} + y\mathbf{j}$ (left) and $\mathbf{F}_2 = \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}}$ (right)

$\mathbf{F}_1$ is the so-called 'radial direction vector,' which is often denoted $\mathbf{r}$. The arrows point in the normal direction to circles centered at the origin and increase in length away from the origin. In three space dimensions, we have

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

The vector $\mathbf{F}_2$ is a unit vector field i.e has constant magnitude one. All the arrows are the same length and point in the tangential direction at the boundaries of circles centered at the origin.

Grad

Given a scalar function, f, recall that we define the gradient ('grad') operator as the vector of first partial derivatives i.e $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$. Applying it to a scalar function gives the vector function:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Div

Given a vector function, $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, the divergence of $\mathbf{F}$ ('div $\mathbf{F}$' for short) is a scalar function defined via:

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

For example, given $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}$, a simple calculation gives $\nabla \cdot \mathbf{F} = 2xy + 0 + xy = 3xy$.

Physically, the divergence of a vector field, at a point (x, y, z) gives a measure of 'net mass flow.' If $\nabla \cdot \mathbf{F} = 0$ then the rate of 'stuff' flowing in is equal to the rate of 'stuff' flowing out. If $\nabla \cdot \mathbf{F} > 0$ then more 'stuff' is flowing out than in etc ...

Exercise

Compute $\nabla \cdot \mathbf{F}_1$ and $\nabla \cdot \mathbf{F}_2$ (from above) and relate your answers to the physical description of divergence.

Curl

The curl of a vector function is defined via:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\
&= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}.
\end{aligned}$$

For example, given $\mathbf{F} = x^2y\mathbf{i} + xz\mathbf{j} + xyz\mathbf{k}$, a simple calculation gives $\nabla \times \mathbf{F} = (xy - x)\mathbf{i} - yz\mathbf{j} + (z - x^2)\mathbf{k}$.

Physically, the curl gives a measure of the twisting or curling of a vector field. If a particle is released into the flow and moves along the flow lines without rotating on its own axis then $\nabla \times \mathbf{F} = 0$. If it spins, $\nabla \times \mathbf{F} \neq 0$.

Exercise

Compute $\nabla \times \mathbf{F}_1$ and $\nabla \times \mathbf{F}_2$ (from above).

Important identities

- For all scalar functions $f = f(x, y, z)$, we have $\nabla \times (\nabla f) = 0$.
- For all vector functions $\mathbf{F}$ we have $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.