Below is a list of some classical partial differential equations (PDEs). They are all derived from basic laws of physics and describe important physical processes that go on in the world around us. You probably recognise the first three on the list from previous courses. In this course, you will learn how to find exact and/or approximate solutions to most of them. You need to become familiar with the names of these equations, be able to classify them as being elliptic, hyperbolic or parabolic, and appreciate what physical processes they each describe. Although it is not necessary to understand how these equations were derived, it helps to understand the physics a little so that we can decide, for example, which boundary and/or initial conditions to apply and so we can interpret solutions in a meaningful way. A brief description of the derivation of the Heat Equation is given below. For more information on the physics behind the PDEs mentioned, consult any engineering text book on PDEs, Haberman's book, Chapters 1 and 4, or even just type the name of the equation into google!

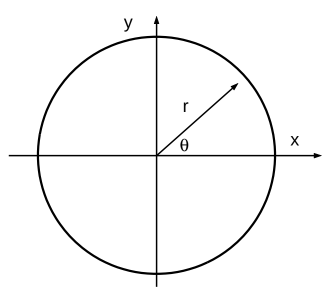
In the list below, u is a function of severable variables and is the solution sought. The independent variables t, x, y and z represent time and the three Cartesian spatial variables, respectively. When we talk about one, two or three-dimensional problems, we refer to the number of dimensions of the spatial domain on which we are solving the PDE. For instance, $u(x,t)$, the solution to the one-dimensional heat equation represents a temperature that varies over time, and with the single spatial variable x in a one-dimensional rod. In two-dimensions, a time-varying temperature $u(x,y,t)$ is sought in a domain that has both x and y dimensions etc. To fully state a PDE problem we also have to include a description of the spatial domain on which the PDE is to be solved, the range of values of t, if the process is time-dependent, and sufficient boundary and initial conditions to ensure that a unique solution exists. If we are interested in solving one of the PDEs below on a circular, cylindrical or spherical domain, then we can convert the equations into polar, cylindrical or spherical coordinates, by mapping each of the derivatives u_x, u_y, u_z into the appropriate coordinate system using the chain rule (see lecture notes).

- **Laplace's Equation:** $\nabla^2 u = 0$

one-dimensional version (ODE):	u_{xx}	$= 0$
two-dimensional version:	$u_{xx} + u_{yy}$	$= 0$
three-dimensional version:	$u_{xx} + u_{yy} + u_{zz}$	$= 0$

Description: Used to model time-independent heat diffusion, (u represents temperature), steady diffusion of liquids through solids (u represents concentration of liquid) and the irrotational flow of an incompressible fluid (u represents a velocity potential).

Boundary/initial conditions: Dirichlet or Neumann boundary conditions are required at all points on the spatial boundary. No initial conditions are needed (time-independent).



Laplace's Equation on a disk: Converting the two-dimensional version of Laplace's Equation, into polar co-ordinates, r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$ yields,

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$

- **Poisson's Equation:** $\nabla^2 u = -f$

$$\begin{array}{lll} \text{one-dimensional version (ODE):} & u_{xx} & = -f(x) \\ \text{two-dimensional version:} & u_{xx} + u_{yy} & = -f(x,y) \\ \text{three-dimensional version:} & u_{xx} + u_{yy} + u_{zz} & = -f(x,y,z) \end{array}$$

Description: Laplace's equation with extra 'source' term f. A typical application is electrostatics where u represents electric potential and f is the ratio of charge density to electric permittivity.
Boundary/initial conditions: Dirichlet or Neumann boundary conditions are required at all points of the spatial boundary. No initial conditions (time-independent).

- **The Heat (Diffusion) Equation:** $u_t = K\nabla^2 u$

$$\begin{array}{lll} \text{one-dimensional version:} & u_t & = Ku_{xx} \\ \text{two-dimensional version:} & u_t & = Ku_{xx} + Ku_{yy} \\ \text{three-dimensional version:} & u_t & = Ku_{xx} + Ku_{yy} + Ku_{zz} \end{array}$$

Description: Used to model time-dependent heat diffusion, (u denotes temperature and K is the thermal conductivity of the material), the diffusion of a liquid through a porous solid, (u denotes concentration of the liquid and K is related to the porosity coefficient of the material.)
Boundary/initial conditions: Dirichlet or Neumann boundary conditions are required at all points on the spatial boundary plus one initial condition.

- **The Wave Equation:** $u_{tt} = c^2\nabla^2 u$

$$\begin{array}{ll} \text{one-dimensional version:} & u_{tt} = c^2 u_{xx} \\ \text{two-dimensional version:} & u_{tt} = c^2\left(u_{xx} + u_{yy}\right) \\ \text{three-dimensional version:} & u_{tt} = c^2\left(u_{xx} + u_{yy} + u_{zz}\right) \end{array}$$

Description: Used to model the propagation of waves (including sound, light and water waves) and vibrations in solids and gases. u represents the deflection or displacement in the object being disturbed and c^2 is the propagation speed of the wave.
Boundary/initial conditions: Dirichlet or Neumann boundary conditions are required at all points on the spatial boundary plus two initial conditions

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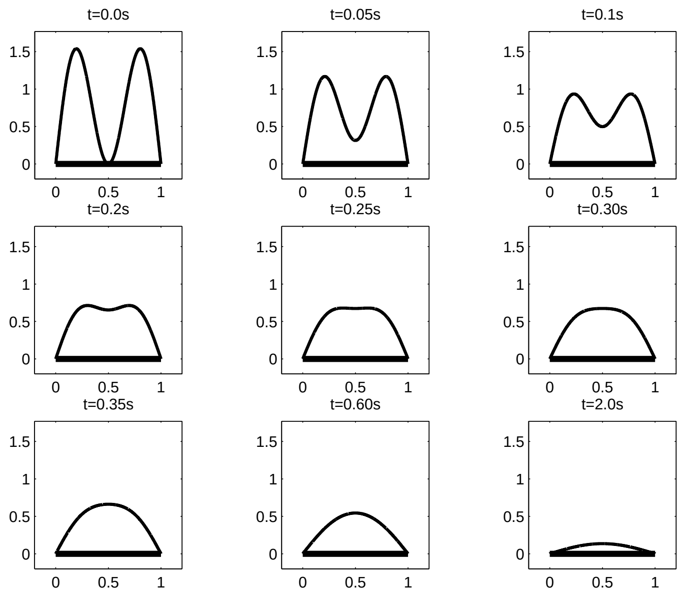
- **The Convection-Diffusion Equation:** $u_t = \vec{w}\cdot\nabla u + \epsilon\nabla^2 u$

$$\begin{array}{ll} \text{one-dimensional version:} & u_t = wu_x + \epsilon u_{xx} \\ \text{two-dimensional version:} & u_t = w_1u_x + w_2u_y + \epsilon\left(u_{xx} + u_{yy}\right) \\ \text{three-dimensional version:} & u_t = w_1u_x + w_2u_y + w_3u_z + \epsilon\left(u_{xx} + u_{yy} + u_{zz}\right) \end{array}$$

Description: In fluid flow modelling, u represents the concentration of a pollutant that spreads though a flowing liquid that is moving at velocity $\vec{w}$. The pollutant spreads out due to diffusion and because it is transported ('convected') by the velocity field of the stream into which it is released. The one-dimensional convection-diffusion equation also provides a model for option pricing in mathematical finance. In that case, u represents the option price, ϵ is related to the so-called 'volatility' and w incorporates the interest rate.

Boundary/initial conditions: Dirichlet or Neumann boundary conditions are required at all points on the spatial boundary plus one initial condition.

Example: Heat Equation. The function $u(x,t) = e^{-t}\sin(\pi x) + e^{-9t}\sin(3\pi x)$ satisfies the one-dimensional heat equation $u_t = u_{xx}$ for all values of x in the domain $(0,1)$. At the end points of the domain, the solution satisfies the **boundary conditions** $u(0,t) = 0$ and $u(1,t) = 0$, and at time $t = 0$, the solution satisfies the **initial condition**: $u(x,0) = \sin(\pi x) + \sin(3\pi x)$. The time-evolution behaviour of the solution is plotted for various times in the pictures below. Can you interpret what is happening physically? (Think of the different graphs as representing frames in a movie).



Example: Wave Equation. The function $u(x,t) = \sin(\pi x)\cos(\pi t)$ satisfies the one-dimensional wave equation $u_{tt} = u_{xx}$ for all values of x in the domain $(0,1)$. At the end points of the domain, the solution satisfies the **boundary conditions** $u(0,t) = 0$ and $u(1,t) = 0$, and at time $t = 0$, the solution satisfies the **two initial conditions**: $u(x,0) = \sin(\pi x)$ and $\frac{\partial u}{\partial t}\,|_{t=0} = 0$. The time-evolution behaviour of the solution is plotted for various times in the pictures below. Can you interpret what is happening physically?