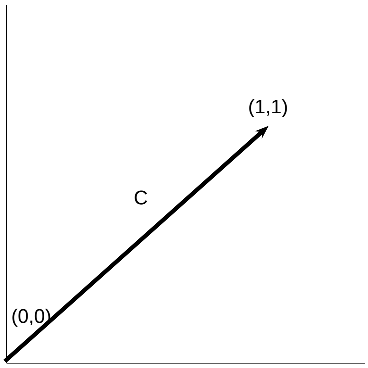
Suppose that we want to integrate some scalar function $f(x,y)$ along a line (a directed curve), c, in two space dimensions (in the $x-y$ plane say), from point a to point b. We can write this as

$$\int_c f(x,y)\,ds$$

where s is the arc length parameter (you met this in 1st year in the course Calculus and Vectors). At a, $s=0$ and at b, s is equal to the length of the line segment $\overrightarrow{ab}$. Each point on the line (x,y) can be parameterized using the variable s as $(x(s),y(s))$. As we vary s from 0 to the length of the line segment, we move along the line from a to b. One way to evaluate the integral is then to express the Cartesian co-ordinates as functions of s and integrate with respect to s. If this can be done, we obtain a standard one-dimensional integral. That is,

$$\int_c f(x,y)\,ds = \int_{s=0}^{\text{length of line}} f(x(s),y(s))\,ds.$$

Example Evaluate $\int_c (x+y)\,ds$ where c is the directed line segment shown below.



The length of the line segment is $\sqrt{2}$ and we have,

$$\int_c (x+y)\,ds = \int_0^{\sqrt{2}} x(s)+y(s)\,ds.$$

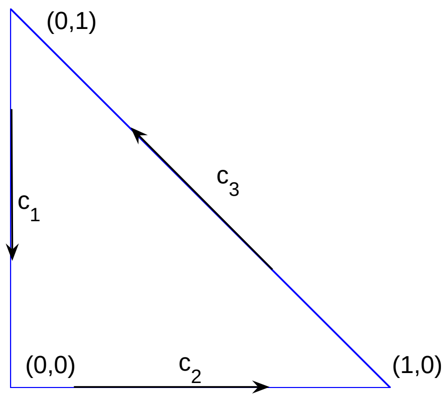
At $(0,0)$, $s=0$ and at $(1,1)$, $s=\sqrt{2}$. The trickiest thing is working out how to express x and y in terms of the variable s. For straight lines, x and y depend linearly on s. So, $x=as+b$ and since $s=0$ at $x=0$, $b=0$. Since $s=\sqrt{2}$ at $x=1$, $a=\frac{1}{\sqrt{2}}$. Hence $x=\frac{s}{\sqrt{2}}$ and similarly $y=\frac{s}{\sqrt{2}}$. Hence,

$$\int_c (x+y)\,ds = \int_0^{\sqrt{2}} 2\frac{s}{\sqrt{2}}\,ds = \sqrt{2}.$$

Recall that the value of such line integrals depends on the path chosen. For example, it we travel from $(0,0)$ to $(1,1)$ by starting at $(0,0)$, move to $(1,0)$ and then move to $(1,1)$, the value of the integral is different. Try it!

Now, if c is a **closed** loop, we usually write $\oint_c$.

Example Evaluate $\oint_c (x+y)\,ds$ where c is the closed loop illustrated below.



We can break the integral over three individual line segments

$$\oint_c (x+y)\,ds = \int_{c_1} (x+y)\,ds + \int_{c_2} (x+y)\,ds + \int_{c_3} (x+y)\,ds.$$

On c_1, changing variables as above gives $x = 0$ and $y = 1 - s$. So $\int_{c_1} (x+y)ds = \int_{s=0}^{1} 1 - s\,ds = \frac{1}{2}$. Similarly, on c_2, $y = 0$ and $x = s$ so $\int_{c_2} (x+y) = \int_{s=0}^{1} s\,ds = \frac{1}{2}$. Finally, on c_3 we have $x = 1 - \frac{s}{\sqrt{2}}$, and $y = \frac{s}{\sqrt{2}}$ so $\int_{c_3} (x+y)\,ds = \int_{s=0}^{\sqrt{2}} 1\,ds = \sqrt{2}$. Therefore, $\oint_c (x+y)ds = \frac{1}{2} + \frac{1}{2} + \sqrt{2} = 1 + \sqrt{2}$.

Further reading If you want to know more about line integrals, read Chapter 3 in the textbook 'Div, grad, curl and all that.'