In the lectures, it was shown that the Fourier Series of the piecewise continuous function

$$f(x) = \begin{cases} 0 & -1 \le x < \frac{1}{2} \\ 1 & \frac{1}{2} < x \le 1 \end{cases}$$

is given by,

$$\frac{1}{4} - \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\pi x) + \sum_{n=1}^{\infty} \frac{1}{n\pi} \left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right] \sin(n\pi x).$$

At first glance, the relationship between $f(x)$ and its Fourier series is not obvious. To get a better feel for what the Fourier Series represents physically, we can simply truncate the series after summing a few terms and plot the result. That is, for a fixed N, we can compute the value of the *finite* series,

$$\frac{1}{4} - \sum_{n=1}^{N} \frac{1}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos(n\pi x) + \sum_{n=1}^{N} \frac{1}{n\pi} \left[\cos\left(\frac{n\pi}{2}\right) - (-1)^n\right] \sin(n\pi x).$$

If we keep increasing N, then we can investigate whether the series converges and whether it converges to the function $f(x)$. For the first few small values of N we obtain,

N	Truncated Fourier Series of $f(x)$
0	$\frac{1}{4}$
1	$\frac{1}{4} - \frac{1}{\pi}\cos(\pi x) + \frac{1}{\pi}\sin(\pi x)$
2	$\frac{1}{4} - \frac{1}{\pi}\cos(\pi x) + \frac{1}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x)$
3	$\frac{1}{4} - \frac{1}{\pi}\cos(\pi x) + \frac{1}{3\pi}\cos(3\pi x) + \frac{1}{\pi}\sin(\pi x) - \frac{1}{\pi}\sin(2\pi x) + \frac{1}{3\pi}\sin(3\pi x)$
4	etc.

We plot the approximations, for each value of N, on the interval $[-1, 1]$ using MATLAB.
For small N, none of these graphs is equal to the graph of $f(x)$ on the interval $[-1, 1]$. As N increases, however, the approximations seem to be converging to a function that 'looks like' $f(x)$. Below we plot the truncated series for $N = 50, 100, 500$ and $1,000$.

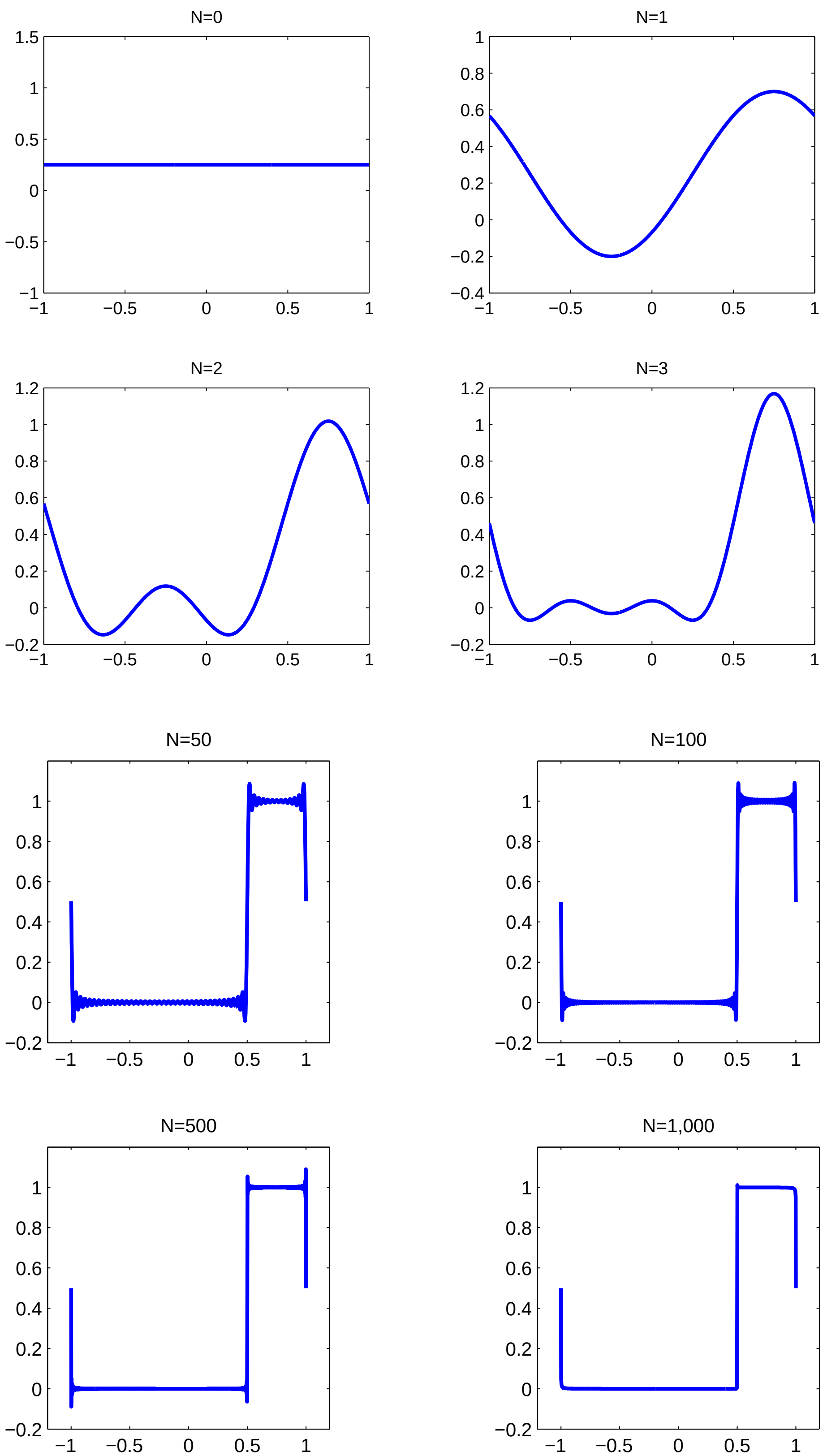
After summing $1,000$ terms, we see that the truncated series has appeared to converge to $f(x)$ on the intervals open $(-1, 0.5)$ and $(0.5, 1)$. However, *at the end-points of the interval, and at the jump point* $x = 0.5$, *the sum of the truncated series we have plotted is not equal to the value of* $f(x)$.

The above graphs were obtained with the following 'naive' MATLAB commands: (try it for different values of N)

```
N=50; f=0.25; x=linspace(-1,1,1000);
for n=1:N
f=f+((-1/(n*pi))*sin((n*pi)/2)*cos(n*pi*x))
+((1/(n*pi))*(cos((n*pi)/2)-(-1)^n)*sin(n*pi*x));
end; plot(x,f);
```

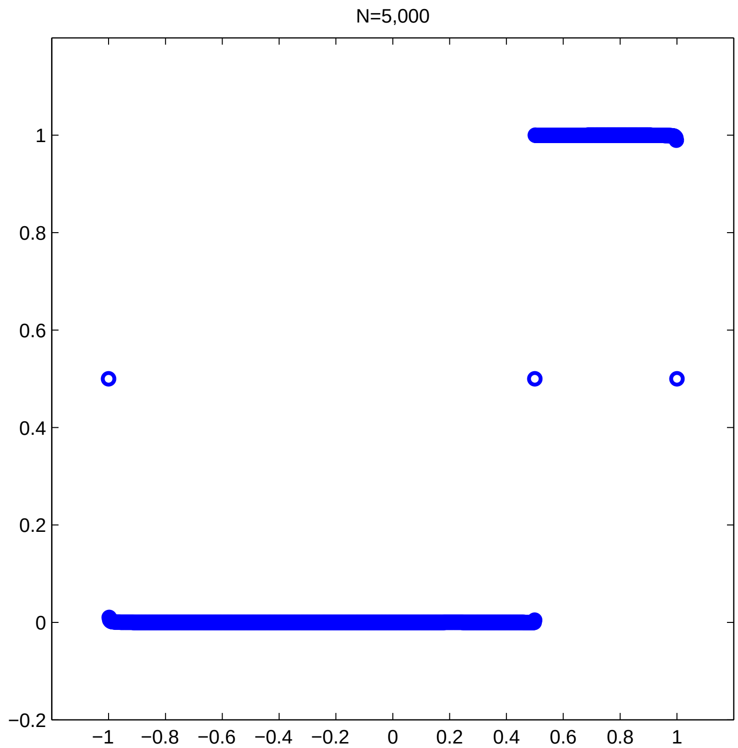
(If you have never used MATLAB, now would be a good time to try it out. Find a computer cluster and look for MATLAB on the programs menu. Start the software and just type the commands above into the command window. I'll bring a laptop to the examples classes to demo this.)

The above code is 'naive' for two reasons. To plot the truncated series, we need to specify a set of values of `x` at which to evaluate the series and then plot the results. If we use the command `x=linspace`$(-1, 1, 1000)$, we tell MATLAB to use 1,000 equally spaced values on the interval $[-1, 1]$. The first value is $x = -1$ and the last value is $x = 1$. In this case, $x = 0.5$ is not one of the chosen x-values. This happens to be a point at which our function $f(x)$ is discontinuous and where interesting things happen! Secondly, when we issue the `plot` command, MATLAB's default is to join up the f-values it has computed, with straight lines. Hence, we do not get the correct picture at $x = -1, 0.5, 1$. We can fix this, however, with two simple changes. We add $x = 0.5$ to the vector of x values and plot using dots rather than joining up the data with lines. The new code is:


N=0
1.5
1
0.5
0
−0.5
−1
−1
−0.5
0
0.5
1
N=1
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−1
−0.5
0
0.5
1
N=2
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1
N=3
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1
N=50
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1
N=100
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1
N=500
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1
N=1,000
1
0.8
0.6
0.4
0.2
0
−0.2
−1
−0.5
0
0.5
1

```
N=50; f=0.25; x=linspace(-1,1,1000); x=[x,0.5];
for n=1:N
f=f+((-1/(n*pi))*sin((n*pi)/2)*cos(n*pi*x))
+((1/(n*pi))*(cos((n*pi)/2)-(-1)^n)*sin(n*pi*x)); end;plot(x,f,'o');
```

Plotting the truncated Fourier Series with $N = 5,000$ with the new code, we obtain the graph below:

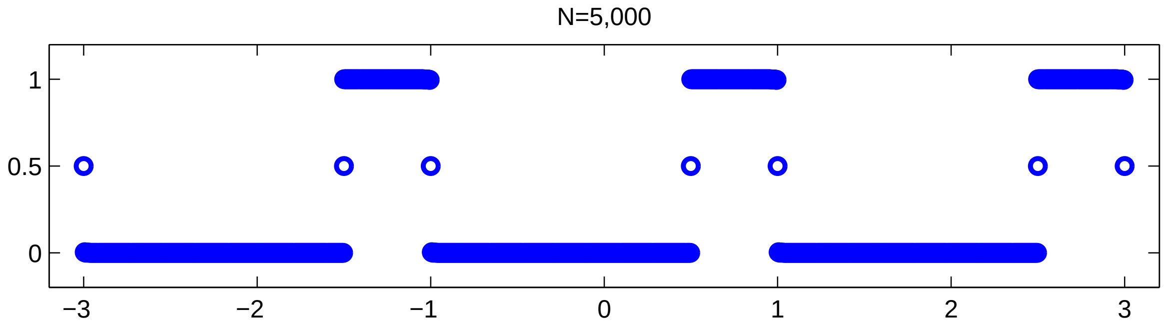


Clearly, we cannot plot the actual Fourier Series because this requires us to sum up infinitely many terms. However, from the approximation plotted above, we can make some interesting conclusions:

- The Fourier Series converges (is equal) to $f(x)$ on the open intervals $(-1, 0.5)$ and $(0.5, 1)$ but *not* on the whole interval $[-1, 1]$.
- At the end points of the interval, $x = -1$ and $x = 1$, the Fourier Series has converged to the value 0.5.
- At the interior jump point $x = 0.5$, the Fourier Series has also converged to the value 0.5.

Why the mysterious value 0.5? What does this correspond to? There are rigorous theoretical results to explain the above phenomena. We shall study them in the lectures.

Notice that in the above graphs, we plotted approximations to the Fourier Series on the interval $[-1, 1]$ since this is the range of values over which our function $f(x)$ is defined. We saw that the Fourier Series of $f(x)$ *is equal to* $f(x)$ *on that interval except at a finite set of three points.* The Fourier Series itself, however, could be plotted over any range of values of x we choose. Below we plot the truncated series with $5,000$ terms for the range $[-3, 3]$.



How is this graph related to $f(x)$? How is it related to the Fourier Series on the principal interval $[-1, 1]$? Why does it exhibit periodic behaviour? What is the period of this function and why? Think about these questions before the next lecture.