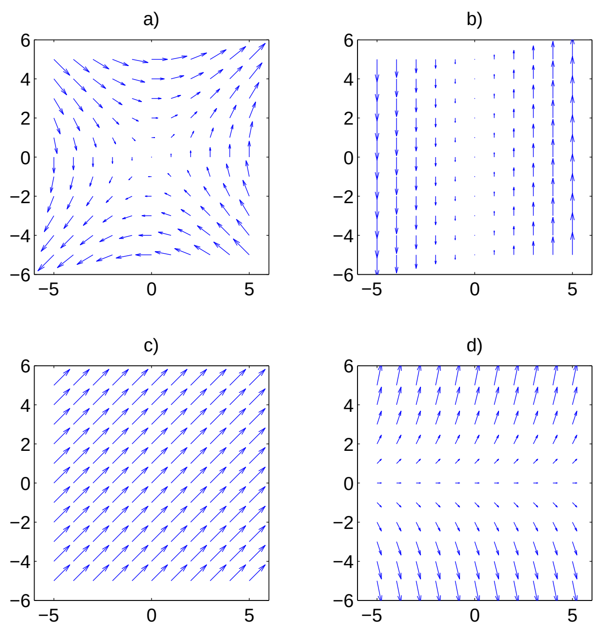
MATH20411 **Problem sheet 9: Vector Calculus A**

1. The following vector fields are plotted in the pictures below.

$$a)\ y\mathbf{i}+x\mathbf{j} \qquad b)\ x\mathbf{j} \qquad c)\ \frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}} \qquad d)\ \mathbf{i}+y\mathbf{j}$$



The lengths of the arrows in each are determined by the modulus
a) $\sqrt{y^2+x^2}$, b) $|x|$, c) 1 (i.e. constant), d) $\sqrt{1+y^2}$.

2. a) The vector that points in the radial direction in two dimensions is $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. However, the magnitude of this vector is $|\mathbf{r}| = \sqrt{x^2+y^2}$. Hence the desired vector is:

$$\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$$

b) In two dimensions the vector has the form $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$. We require $\tan^{-1}\left(\frac{F_y}{F_x}\right) = 45^o$ and hence $F_y = F_x$. For the magnitude we require then that $\sqrt{2F_x^2} = (x+y)^2$, i.e., $F_x = F_y = \frac{(x+y)^2}{\sqrt{2}}$.

c) The vector that points in the radial direction in three dimensions is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. However, the magnitude of this vector is $|\mathbf{r}| = \sqrt{x^2+y^2+z^2}$. Hence the desired vector is:

$$\mathbf{F} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{x^2+y^2+z^2}}$$

3. Given $\mathbf{F} = F_x(x,y,z)\mathbf{i} + F_y(x,y,z)\mathbf{j} + F_z(x,y,z)\mathbf{k}$, applying the curl operator first gives

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}.$$

Applying the definition of the divergence operator to this vector field gives

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \\ &= \frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial x \partial z} + \frac{\partial^2 F_x}{\partial y \partial z} - \frac{\partial^2 F_z}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z \partial y} = 0\end{aligned}$$

(since the derivative operators are commutative).

4. Given any scalar function $f(x,y,z)$, the gradient vector is:

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Applying the definition of the curl operator gives:

$$\nabla \times (\nabla f) = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right)\mathbf{k} = 0.$$

5. The divergences are

$$a)\ 2x + 2y + 2z, \quad b)\ 0, \quad c)\ 3, \quad d)\ \frac{xy - xy}{(x^2+y^2)^{\frac{3}{2}}} = 0.$$

6. The curls are

$$a)\ -2y\mathbf{i} + 2z\mathbf{j} + 2x\mathbf{k}, \quad b)\ 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0, \quad c)\ z\mathbf{i} - x\mathbf{k}, \quad d)\ 0.$$

7. (Revision)

a)

$$\int_{x=0}^{1}\int_{y=0}^{x^3} 3\,dydx = \int_0^1 [3y]_0^{x^3}\,dx = \int_0^1 3x^3\,dx = \left[\frac{3x^4}{4}\right]_0^1 = \frac{3}{4}.$$

b)

$$\int_{y=0}^{2}\int_{x=0}^{y^2} y\,dxdy = \int_0^2 [xy]_0^{y^2}\,dy = \int_0^2 y^3\,dy = \left[\frac{y^4}{4}\right]_0^2 = 4.$$

8. (Revision) Sketch each region of integration in the x-y plane to understand how the limits need to be interchanged.

a)

$$\int_{x=0}^{1}\int_{y=0}^{x} 2 - x - y\,dydx = \int_0^1 \left[2y - xy - \frac{y^2}{2}\right]_0^x dx = \int_0^1 2x - x^2 - \frac{x^2}{2}\,dx = \left[x^2 - \frac{x^3}{3} - \frac{x^3}{6}\right]_0^1 = \frac{1}{2}.$$

$$\int_{y=0}^{1}\int_{x=y}^{1} 2 - x - y\,dxdy = \int_0^1 \left[2x - \frac{x^2}{2} - xy\right]_y^1 dy = \int_0^1 \frac{3}{2} - 3y + \frac{3y^2}{2}\,dy = \left[\frac{3y}{2} - \frac{3y^2}{2} + \frac{y^3}{2}\right]_0^1 = \frac{1}{2}.$$

b)

$$\int_{x=0}^{3}\int_{y=1}^{e^x} 2\,dydx = \int_0^3 [2y]_1^{e^x}\;dx = \int_0^3 2e^x - 2\,dx = [2e^x - 2x]_0^3 = 2e^3 - 8.$$

$$\int_{y=1}^{e^3}\int_{x=\ln y}^{3} 2\,dxdy = \int_1^{e^3} [2x]_{\ln y}^3\;dx = \int_1^{e^3} 6 - 2\ln y\,dy = [8y - 2y\ln y]_1^{e^3} = 2e^3 - 8.$$

c)

$$\int_{x=0}^{\frac{\pi}{2}}\int_{y=0}^{\cos(x)} \sin(x)\;dydx = \int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin(2x)\,dx = \left[-\frac{1}{4}\cos(2x)\right]_{\frac{\pi}{2}} = \frac{1}{2}.$$

$$\int_{y=0}^{1}\int_{x=0}^{\cos^{-1}(y)} \sin(x)\,dxdy = \int_0^1 -y + 1\,dy = \left[-\frac{y^2}{2} + y\right]_0^1 = \frac{1}{2}.$$

9.

$$\begin{aligned}\int\int\int_V x^2e^y + xyzdV &= \int_{x=-2}^{3}\int_{y=0}^{1}\int_{z=0}^{5}\left(x^2e^y + xyz\right)dz\,dy\,dx = \int_{x=-2}^{3}\int_{y=0}^{1}\left[zx^2e^y + \frac{xyz^2}{2}\right]_0^5 dy\,dx \\ &= \int_{x=-2}^{3}\int_{y=0}^{1} 5x^2e^y + \frac{25}{2}xydy\,dx = \int_{x=-2}^{3}\left[5x^2e^y + \frac{25xy^2}{4}\right]_0^1 dx \\ &= \int_{x=-2}^{3} 5x^2e + \frac{25x}{4} - 5x^2\,dx = \left[\frac{5x^3e}{3} + \frac{25x^2}{8} - \frac{5x^3}{3}\right]_{-2}^{3} = \frac{175}{3}(e-1) + \frac{125}{8}\end{aligned}$$

10. a)

$$\begin{aligned}\int_{x=-1}^{1}\int_{y=0}^{2}\int_{z=1}^{3} xyzdz\,dy\,dx &= \int_{x=-1}^{1}\int_{y=0}^{2}\left[\frac{xyz^2}{2}\right]_1^3 dy\,dx = \int_{x=-1}^{1}\int_{y=0}^{2} 4xy\,dy\,dx \\ &= \int_{x=-1}^{1}\left[2xy^2\right]_0^2\,dx = \int_{x=-1}^{1} 8x\,dx = \left[4x^2\right]_{-1}^{1} = 0.\end{aligned}$$

b)

$$\begin{aligned}\int_{x=0}^{1}\int_{y=0}^{2}\int_{z=0}^{3} x^2 + y^2 + z^2dz\,dy\,dx &= \int_{x=0}^{1}\int_{y=0}^{2}\left[zx^2 + zy^2 + \frac{z^3}{3}\right]_0^3 dy\,dx = \int_{x=0}^{1}\int_{y=0}^{2} 3x^2 + 3y^2 + 9\,dy\,dx \\ &= \int_{x=0}^{1}\left[3x^2y + y^3 + 9y\right]_0^2\,dx = \int_{x=0}^{1} 6x^2 + 26\,dx = \left[2x^3 + 26x\right]_0^1 = 28.\end{aligned}$$

c)

$$\begin{aligned}\int_{z=-1}^{2}\int_{y=1}^{z^2}\int_{x=0}^{y+z} 3yz^2dx\,dy\,dz &= \int_{z=-1}^{2}\int_{y=1}^{z^2}\left[3yz^2x\right]_0^{y+z} dy\,dx = \int_{z=-1}^{2}\int_{y=1}^{z^2} 3y^2z^2 + 3yz^3\,dy\,dz \\ &= \int_{z=-1}^{2}\left[y^3z^2 + \frac{3y^2z^3}{2}\right]_1^{z^2} dx = \int_{z=-1}^{2} z^8 + \frac{3}{2}z^7 - z^2 - \frac{3}{2}z^3\,dx \\ &= \left[\frac{z^9}{9} + \frac{3z^8}{16} - \frac{z^3}{3} - \frac{3z^4}{8}\right]_{-1}^{2}\end{aligned}$$

11. The size of the volume $|V|$ is given by the integral $\int\int\int_V 1dxdydz$ where V is the volume enclosed by the cylinder. Changing to cylindrical coordinates for simplicity gives:

$$
\begin{aligned}
|V| = \int\int\int_V 1dxdydz &= \int\int\int_V 1rdrd\theta dz = \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\int_{r=0}^{a} rdrd\theta dz \\
&= \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_0^a d\theta dz = \int_{z=0}^{b}\int_{\theta=0}^{2\pi}\frac{a^2}{2}d\theta dz \\
&= \int_{z=0}^{b}\left[\theta\frac{a^2}{2}\right]_0^{2\pi} dz = \int_{z=0}^{b}\pi a^2 dz = \left[z\pi a^2\right]_0^b = \pi a^2 b.
\end{aligned}
$$