1. First note that using separation of variables we obtain infinitely many solutions to the heat equation with homogenous Dirichlet boundary conditions of the form: $u_n(x,t) = C_n e^{-(n\pi)^2 t}\sin(n\pi x)$ where C_n is an arbitrary constant and $n = 1, 2, \ldots$. To impose the initial condition we apply the principle of superposition (or just note that only the coefficients corresponding to $n = 1$ and $n = 3$ are non-zero) to obtain: $u(x,t) = e^{-\pi^2 t}\sin(\pi x) + e^{-9\pi^2 t}\sin(3\pi x)$.

   Now, for the approximation, set $N = 6$ intervals on $[0,1]$ and set $x_0 = 0$, $x_1 = \frac{1}{6}$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{2}$ and $x_4 = \frac{2}{3}$, $x_4 = \frac{5}{6}$, $x_6 = 1$. Let U_j^m denote an approximation to the exact value $u(x_j, t_m)$. If we set $t_0 = 0$ then the explicit finite difference scheme based on centered differences in space and a forward difference in time (see lecture notes) yields 5 equations for approximations to $u(x,t)$ at the interior space nodes, at each new level t_m. We have:

$$U_j^{m+1} = \nu U_{j-1}^m + (1-2\nu)\,U_j^m + \nu U_{j+1}^m, \quad j = 1:5, \quad m = 0,1,2,\ldots$$

   where $\nu = \frac{k}{h^2}$. The boundary conditions give values for the end points at each time level:

$$U_0^m = U_6^m = 0, \quad m = 0,1,2,\ldots$$

   With $h = \frac{1}{6}$, we obtain five equations for the unknown values $U_1^{m+1}, U_2^{m+1}, U_3^{m+1}, U_4^{m+1}, U_5^{m+1}$ at each new time step:

$$\begin{array}{cccccc}
(1-72k)\,U_1^m & +36kU_2^m & & & & = U_1^{m+1}\\
36kU_1^m & +(1-72k)\,U_2^m & +36kU_3^m & & & = U_2^{m+1}\\
 & +36kU_2^m & +(1-72k)\,U_3^m & +36kU_4^m & & = U_3^{m+1}\\
 & & 36kU_3^m & +(1-72k)\,U_4^m & +36u_5^m & = U_4^{m+1}\\
 & & & 36kU_4^m & +(1-72k)\,U_5^m & = U_5^{m+1}
\end{array}$$

$$\Rightarrow \begin{pmatrix} 1-72k & 36k & 0 & 0 & 0\\ 36k & 1-72k & 36k & 0 & 0\\ 0 & 36k & 1-72k & 36k & 0\\ 0 & 0 & 36k & 1-72k & 36k\\ 0 & 0 & 0 & 36k & 1-72k \end{pmatrix}\begin{pmatrix} U_1^m\\ U_2^m\\ U_3^m\\ U_4^m\\ U_5^m\end{pmatrix} = \begin{pmatrix} U_1^{m+1}\\ U_2^{m+1}\\ U_3^{m+1}\\ U_4^{m+1}\\ U_5^{m+1}\end{pmatrix}.$$

   If we set $t_0 = 0$ and choose $k = 0.002$ and notice that the initial condition gives:

$$U_1^0 = \sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{2}\right) = 1.5, \quad U_2^0 = \sin\left(\frac{\pi}{3}\right) + \sin(\pi) = \frac{\sqrt{3}}{2}, \quad U_3^0 = 0, U_4^0 = \frac{\sqrt{3}}{2}, U_5^0 = 1.5$$

   Substituting these values into the finite difference formulae with $k = 0.002$ yields (to 6 d.p.):

$$\begin{array}{rcccl}
U_1^1 & = & 0.856\,(1.5) + 0.072\frac{\sqrt{3}}{2} & = & 1.346354\\
U_2^1 & = & 0.072\,(1.5) + 0.856\frac{\sqrt{3}}{2} + 0.072\,(0) & = & 0.849318\\
U_3^1 & = & 0.072\left(\frac{\sqrt{3}}{2}\right) + 0.856\,(0) + 0.072\left(\frac{\sqrt{3}}{2}\right) & = & 0.124708\\
U_4^m & = & 0.072\,(0) + 0.856\left(\frac{\sqrt{3}}{2}\right) + 0.072\,(1.5) & = & 0.849318\\
U_5^1 & = & 0.072\left(\frac{\sqrt{3}}{2}\right) + 0.856\,(1.5) & = & 1.346354
\end{array}$$

The errors at the boundary points x_0 and x_6 are zero since the numerical solution agrees with the exact solutions there. At the interior points, we have:

$$e_1^1 = U_1^1 - u\left(\frac{1}{6}, 0.002\right) = 1.346354 - 1.327460 = 0.018894,$$

$$e_2^1 = 0.000219,\ e_3^1 = -0.0185137,\ e_4^1 = 0.000219,\ e_5^1 = 0.000219.$$

You can sketch the exact and approximate solution at the grid points at $t = t_1 = 0.002$ by hand or using MATLAB (whichever you prefer). MATLAB plots are given below.

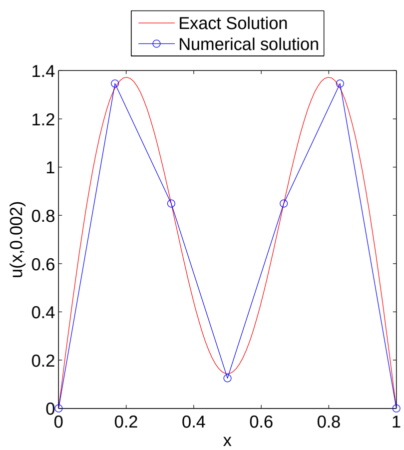

Figure 1: Exact solution and numerical approximation to the solution after one time step

2. The truncation error at a point (x_j, t_m) is the remainder when the exact solution is substituted into the numerical scheme, i.e.:

$$T_j^m = \frac{u(x_j, t_{m+1}) - u(x_j, t_m)}{k} - \left(\frac{u(x_{j+1}, t_m) - 2u(x_j, t_m) + u(x_{j-1}, t_m)}{h^2}\right).$$

Expanding the terms $u(x_j, t_{m+1})$, $u(x_{j+1}, t_m)$ and $u(x_{j-1}, t_m)$ about the point (x_j, t_m) using Taylor series expansions gives:

$$u(x_j, t_{m+1}) = u(x_j, t_m) + k u_t(x_j, t_m) + \frac{(k)^2}{2} u_{tt}(x_j, t_m) + \frac{(k)^3}{6} u_{ttt}(x_j, t_m) + \dots$$

$$u(x_{j+1}, t_m) = u(x_j, t_m) + h u_x(x_j, t_m) + \frac{h^2}{2} u_{xx}(x_j, t_m) + \frac{h^3}{6} u_{xxx}(x_j, t_m) + \frac{h^4}{24} u_{xxxx}(x_j, t_m) + \dots$$

$$u(x_{j-1}, t_m) = u(x_j, t_m) - h u_x(x_j, t_m) + \frac{h^2}{2} u_{xx}(x_j, t_m) - \frac{h^3}{6} u_{xxx}(x_j, t_m) + \frac{h^4}{24} u_{xxxx}(x_j, t_m) + \dots$$

Substituting these expressions into the truncation error and rearranging gives:

$$T_j^m = u_t(x_j, t_m) - u_{xx}(x_j, t_m) + \frac{(k)}{2} u_{tt}(x_j, t_m) - \frac{h^2}{12} u_{xxxx}(x_j, t_m) + \dots$$

The sum of the first two terms is zero (from the definition of the differential equation). Rearranging the other terms by writing $\nu = \frac{k}{h^2}$ gives,

$$T_j^m = \frac{k}{2}\left(u_{tt}\left(x_j, t_m\right) - \frac{1}{6\nu}u_{xxxx}\left(x_j, t_m\right)\right) + \text{terms with higher powers of } k.$$

Since the leading term depends on k for a fixed value of ν, we see that the scheme is first order accurate w.r.t k.

3. The numerical scheme gives:

$$\frac{U_j^{m+1} - U_j^m}{k} - \frac{\left(U_{j+1}^m - 2U_j^m + U_{j-1}^m\right)}{h^2} = 0, \quad j = 1:N-1,\ m = 1, 2, \ldots$$

and the exact solution satisfies:

$$\frac{u\left(x_j, t_{m+1}\right) - u\left(x_j, t_m\right)}{k} - \left(\frac{u\left(x_{j+1}, t_m\right) - 2u\left(x_j, t_m\right) + u\left(x_{j-1}, t_m\right)}{h^2}\right) = T_j^m, \quad j = 1:N-1,\ m = 1, 2, \ldots$$

Subtracting gives

$$\frac{e_j^{m+1} - e_j^m}{k} - \left(\frac{e_{j+1}^m - 2e_j^m + e_{j-1}^m}{h^2}\right) = T_j^m, \quad j = 1:N-1,\ m = 1, 2, \ldots$$

Rearranging and writing $\nu = \frac{k}{h^2}$ gives,

$$e_j^{m+1} = \nu e_{j-1}^m + (1 - 2\nu)e_j^m + \nu e_{j+1}^m + kT_j^m, \quad j = 1:N-1,\ m = 1, 2, \ldots$$

Now, consider taking the absolute value of both sides of this equation.

$$\mid e_j^{m+1} \mid \leq \mid \nu e_{j-1}^m + (1 - 2\nu)e_j^m + \nu e_{j+1}^m \mid + \mid kT_j^m \mid, \quad j = 1:N-1,\ m = 1, 2, \ldots$$

The key point is that if $\nu \leq \frac{1}{2}$ the coefficient $(1 - 2\nu)$ is not negative and so:

$$\mid e_j^{m+1} \mid \leq \nu \mid e_{j-1}^m \mid + (1 - 2\nu) \mid e_j^m \mid + \nu \mid e_{j+1}^m \mid + k \mid T_j^m \mid, \quad j = 1:N-1,\ m = 1, 2, \ldots$$

Now, if E^m denotes the maximum error over all grid points at time level t_m, we have $\mid e_{j-1}^m \mid \leq E^m$, $\mid e_j^m \mid \leq E^m$ and $\mid e_{j+1}^m \mid \leq E^m$, giving:

$$\mid e_j^{m+1} \mid \leq (\nu + 1 - 2\nu + \nu)\, E^m + k \mid T_j^m \mid \quad i = 1:N = 1,\ m = 1, 2, \ldots$$

or,

$$\mid e_j^{m+1} \mid \leq E^m + k \mid T_j^m \mid \quad j = 1:N = 1,\ m = 1, 2, \ldots$$

Now, taking the maximum over $j = 1:N-1$ at each time level gives:

$$E^{m+1} = \max_j \mid e_j^{m+1} \mid \leq E^m + k \max_j \mid T_j^m \mid \quad m = 1, 2, \ldots$$

Given that an upper-bound T^* exists for the truncation error T^m at each time level, we have,

$$E^{m+1} \leq E^m + kT^* \quad m = 1, 2, \ldots$$

Since the approximation agrees with the exact solution at all the grid points at $t = t_0$, we have $E_0 = 0$, $E^1 \leq kT^*$, $E^2 \leq E^1 + kT^* \leq 2kT^*$ and, in general, at the mth time step $E^m \leq mkT^* = t_m T^*$.

4. Suppose we choose $N = 6$ intervals on $[0,1]$ and set $x_0 = 0$, $x_1 = \frac{1}{6}$, $x_2 = \frac{1}{3}$, $x_3 = \frac{1}{2}$ and $x_4 = \frac{2}{3}, x_5 = \frac{5}{6}, x_6 = 1$. Let U_j^m denote an approximation to the exact solution $u(x_j, t_m)$. If we set $t_0 = 0$ then the implicit finite difference scheme based on centered differences in space and a backward difference in time (see lecture notes) yields 5 equations for approximations to $u(x,t)$ at the interior space nodes, at each new level t_m. We have:

$$U_j^m = -\nu U_{j-1}^{m+1} + (1+2\nu)\, U_j^{m+1} - \nu U_{j+1}^{m+1}, \quad j = 1:5, \quad m = 1, 2, \ldots$$

where $\nu = 36k$. The boundary conditions give values for the end points at each time level:

$$U_0^m = U_6^m = 0, \quad m = 0, 1, 2, \ldots$$

With $h = \frac{1}{6}$ and $k = 0.01$ we obtain 5 equations for the unknown values $U_1^{m+1}, U_2^{m+1}, U_3^{m+1}, U_4^{m+1}, U_5^{m+1}$ at each new time step:

$$\begin{array}{rrrrrcl}
1.72U_1^{m+1} & -0.36kU_2^{m+1} & & & & = & U_1^m \\
-0.36kU_1^{m+1} & +1.72U_2^{m+1} & -0.36U_3^{m+1} & & & = & U_2^m \\
 & -0.36U_2^{m+1} & +1.72U_3^{m+1} & & & = & U_3^m \\
 & & -0.36U_3^{m+1} & +1.72U_4^{m+1} & -0.36U_5^{m+1} & = & U_4^m \\
 & & & -0.36U_4^{m+1} & +1.72U_5^{m+1} & = & U_5^m
\end{array}$$

$$\Rightarrow \begin{pmatrix} 1.72 & -0.36 & 0 & 0 & 0 \\ -0.36 & 1.72 & -0.36 & 0 & 0 \\ 0 & -0.36 & 1.72 & -0.36 & 0 \\ 0 & 0 & -0.36 & 1.72 & -0.36 \\ 0 & 0 & 0 & -0.36 & 1.72 \end{pmatrix} \begin{pmatrix} U_1^{m+1} \\ U_2^{m+1} \\ U_3^{m+1} \\ U_4^{m+1} \\ U_5^{m+1} \end{pmatrix} = \begin{pmatrix} U_1^m \\ U_2^m \\ U_3^m \\ U_4^m \\ U_5^m \end{pmatrix}.$$

Setting $t_0 = 0$ and noticing that the initial condition $f(x) = \sin(\pi x)$ gives:

$$U_1^0 = f(x_1) = \sin\left(\frac{\pi}{6}\right) = 0.5, \quad U_2^0 = \frac{\sqrt{3}}{2}, \quad U_3^0 = 1, \quad U_4^0 = \frac{\sqrt{3}}{2}, \quad U_5^0 = 0.5$$

we then have to solve a 5×5 linear system (by hand or using MATLAB) for the solution at $t_1 = 0.01s$ ie `u1=trisolve(5,-0.36,1.72,-0.36,[1/2;sqrt(3)/2;1;sqrt(3)/2; 1/2])`.

$$\begin{pmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \\ U_4^1 \\ U_5^1 \end{pmatrix} = \begin{pmatrix} 0.4560 \\ 0.7898 \\ 0.9120 \\ 0.7898 \\ 0.4560 \end{pmatrix}.$$

Note that the exact solution (see question one) is $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$. The exact values (to 4 d.p) at the grid points at $t = 0.001$ are `[0.4530;0.7846; 0.9060; 0.7846;0.4530]`.

You can sketch the exact and approximate solution at the grid points at $t = t_1 = 0.001$ by hand or using MATLAB (whichever you prefer). MATLAB plots are given below.

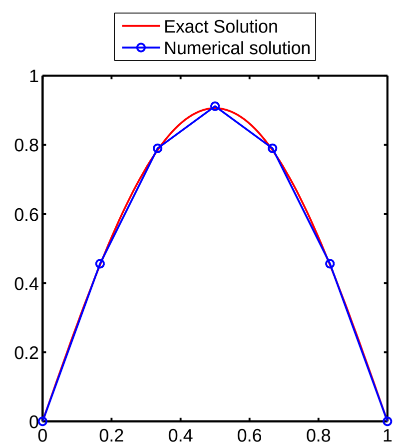

Figure 2: Exact solution and numerical approximation to the solution after one time step