1. The first centered difference is $\delta u(x_j) = u\left(x_j + \frac{h}{2}\right) - u\left(x_j - \frac{h}{2}\right)$. Applying the definition twice yields

$$\delta^2 u(x_j) = \delta\left(\delta u(x_j)\right) = \left(u\left(x_j + \frac{h}{2} + \frac{h}{2}\right) - u\left(x_j - \frac{h}{2} + \frac{h}{2}\right)\right) - \left(u\left(x_j + \frac{h}{2} - \frac{h}{2}\right) - u\left(x_j - \frac{h}{2} - \frac{h}{2}\right)\right)$$

$$= u\left(x_j + h\right) - 2u(x_j) + u\left(x_j - h\right) = u\left(x_{j+1}\right) - 2u(x_j) + u\left(x_{j-1}\right)$$

Note that $\frac{1}{h}\delta u(x_j)$ is the gradient of the line joining $\left(x_j + \frac{h}{2}, u\left(x_j + \frac{h}{2}\right)\right)$ and $\left(x_j - \frac{h}{2}, u\left(x_j - \frac{h}{2}\right)\right)$ and so gives an approximation to the first derivative of u at $x = x_j$, i.e., $\frac{1}{h}\delta u(x_j) \approx \frac{du}{dx}(x_j)$. Hence:

$$\frac{d^2u}{dx^2}(x_j) = \frac{d}{dx}\left(\frac{du}{dx}(x_j)\right) \approx \frac{1}{h}\delta\left(\frac{1}{h}\delta u(x_j)\right) = \frac{1}{h^2}\delta^2 u(x_j).$$

2. a) With $r(x) = 0$, $f(x) = 1$, the reaction-diffusion equation reduces to,

$$-\frac{d^2u}{dx^2} = 1, \qquad 0 < x < 1, \text{ such that} \quad u(0) = 0,\, u(1) = 0.$$

The exact solution is $u(x) = \frac{1}{2}\left(x - x^2\right)$ [check!]. With $N = 5$ intervals, we have $h = \frac{1}{5}$ so $x_0 = 0$, $x_1 = \frac{1}{5}$, $x_2 = \frac{2}{5}$, $x_3 = \frac{3}{5}$, $x_4 = \frac{4}{5}$, $x_5 = 1$. Writing $U_j \approx u(x_j)$, and replacing the second derivative by the second centered difference operator defined in question one gives:

$$-\frac{1}{h^2}U_{j-1} + \frac{2}{h^2}U_j - \frac{1}{h^2}U_{j+1} = 1, \qquad j = 1:4, \quad U_0 = 0,\, U_5 = 0.$$

Substituting in for h gives,

$$-25U_{j-1} + 50U_j - 25U_{j+1} = 1, \qquad j = 1:4, \quad U_0 = 0,\, U_5 = 0.$$

In matrix notation we have,

$$\begin{pmatrix} 50 & -25 & 0 & 0 \\ -25 & 50 & -25 & 0 \\ 0 & -25 & 50 & -25 \\ 0 & 0 & -25 & 50 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Using the MATLAB code `trisolve.m` the solution is given by:

```
u = trisolve(4,-25,50,-25,[1;1;1;1]).
```

The first input argument is the number of equations, the second to fourth inputs are the diagonal entries of the tridiagonal matrix and the final input is the right-hand side vector. The solution is: $U_1 = 0.0800 = \frac{2}{25}$, $U_2 = 0.1200 = \frac{3}{25}$, $U_3 = 0.1200 = \frac{3}{25}$, $U_4 = 0.0800 = \frac{2}{25}$. Comparing with the exact solution $u(x) = \frac{1}{2}\left(x - x^2\right)$, we obtain, $u(\frac{1}{5}) = \frac{2}{25}$, $u(\frac{2}{5}) = \frac{3}{25}$,$u(\frac{3}{5}) = \frac{3}{25}$, $u(\frac{4}{5}) = \frac{2}{25}$. Hence the approximation is EXACT at each of the grid points.

To check your computations and plot the graph above, run the code `reac_diff_1d` with $w = 0$ and $N = 5$ via the command:

```
[u,u_exactx,error] = reac_diff_1D(0,5).
```

b) With $r(x) = 16$, $f(x) = 1$, the reaction-diffusion equation is,

$$-\frac{d^2u}{dx^2} + 16u = 1, \qquad 0 < x < 1, \text{ such that} \quad u(0) = 0,\, u(1) = 0.$$

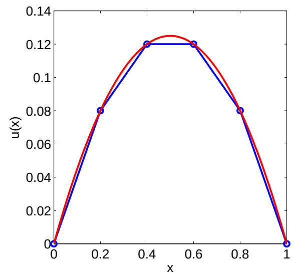

Figure 1: Exact solution (solid line) and approximation with 5 intervals (solid line with circles)

The exact solution (given in lectures) is:

$$u(x) = \frac{1}{16} - \frac{(\exp(4x) + \exp(4(1-x)))}{16(1+\exp(4))}.$$

With $N = 5$ intervals, we have $h = \frac{1}{5}$ so $x_0 = 0$, $x_1 = \frac{1}{5}$, $x_2 = \frac{2}{5}$, $x_3 = \frac{3}{5}$, $x_4 = \frac{4}{5}$, $x_5 = 1$. Writing $U_j \approx u(x_j)$, and replacing the second derivative by the second centered difference operator defined in question one gives:

$$-\frac{1}{h^2}U_{j-1} + \left(\frac{2}{h^2} + 16\right)U_j - \frac{1}{h^2}U_{j+1} = 1, \qquad j = 1:4, \quad U_0 = 0, U_5 = 0.$$

Substituting in for h gives,

$$-25U_{j-1} + 66U_j - 25U_{j+1} = 1, \qquad j = 1:4, \quad U_0 = 0, U_5 = 0.$$

In matrix notation we have,

$$\begin{pmatrix} 66 & -25 & 0 & 0 \\ -25 & 66 & -25 & 0 \\ 0 & -25 & 66 & -25 \\ 0 & 0 & -25 & 66 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Using the MATLAB code `trisolve.m` the solution is given by:

```
u = trisolve(4,-25,66,-25,[1;1;1;1]).
```

The solution is: $U_1 = 0.0317$ $U_2 = 0.0437$ $U_3 = 0.0437$ $U_4 = 0.0317$. Comparing with the exact solution we obtain, $u(\frac{1}{5}) = 0.0324$ $u(\frac{2}{5}) = 0.0445$, $u(\frac{3}{5}) = 0.0445$, $u(\frac{4}{5}) = 0.0324$. Hence the approximation is NOT EXACT at each of the grid points.
To check your computations and plot the graph above, run the code `reac_diff_1d` with $w = 4$ and $N = 5$ via the command:

```
[u,u_exactx,error]=reac_diff_1d(4,5).
```

3. From question 1 we have the approximation:

$$\frac{d^2u}{dx^2}(x_j) \approx \frac{1}{h^2}\delta^2 u(x_j) = \frac{1}{h^2}\left(u(x_j+h) - 2u(x_j) + u(x_j-h)\right).$$

Expanding about $x = x_j$ using Taylor series gives:

$$\begin{aligned} u(x_j+h) &= u(x_j) + hu'(x_j) + \frac{h^2}{2}u''(x_j) + \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(4)}(x_j) \\ &+ \text{ terms with higher powers of h and higher derivatives of } u \\ u(x_j-h) &= u(x_j) - hu'(x_j) + \frac{h^2}{2}u''(x_j) - \frac{h^3}{6}u'''(x_j) + \frac{h^4}{24}u^{(4)}(x_j) \\ &+ \text{ terms with higher powers of h and higher derivatives of } u \end{aligned}$$

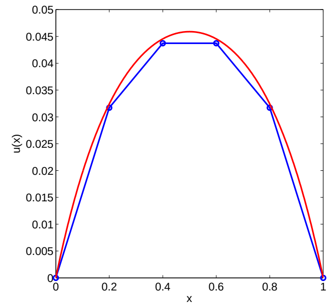

Figure 2: Exact solution (solid line) and approximation with 5 intervals (solid line with circles)

Substituting into the approximation gives:

$$\frac{1}{h^2}\delta^2 u(x_j) = \frac{1}{h^2}\left(h^2 u''(x_j) + \frac{2h^4}{24}u^{(4)}(x_j) + \ldots\right).$$

Hence,

$$\frac{1}{h^2}\delta^2 u\left(x_j\right) - \frac{d^2 u\left(x_j\right)}{dx^2} = \frac{h^2}{12}\frac{d^4 u(x_j)}{dx^4} + \text{terms with higher powers of } h. \tag{1}$$

Note that for question 2 a), the exact solution was $u(x) = \frac{1}{2}\left(x - x^2\right)$. For such a function, the fourth and higher derivatives are all zero, hence the centered difference is an exact representation of the second derivative at the points $x = x_j$.

The truncation error at x_j is the remainder when the exact solution is substituted into the numerical approximation scheme. In this case, we have:

$$T_j = -\frac{1}{h^2}\delta^2 u\left(x_j\right) - r(x_j)u(x_j) - f(x_j).$$

Since we also know that the exact solution at x_j satisfies the differential equation exactly we have:

$$0 = -\frac{d^2 u(x_j)}{dx^2} - r(x_j)u(x_j) - f(x_j).$$

Subtracting these two expressions tells us that:

$$T_j = \frac{d^2 u\left(x_j\right)}{dx^2} - \frac{1}{h^2}\delta^2 u\left(x_j\right).$$

Using (1) then gives the desired result.

Thus $|T_j| = 0$ for all x_j in part a). In other words, no information is lost by replacing the second derivative by the finite difference approximation. In part b), the fourth derivative of the exact solution is not zero and the truncation error T_j is not zero. Information is now lost by replacing the second derivative in the PDE with the finite difference.

In fact, it can be shown that the global error at a point x_j, given by $e_j = u(x_j) - U_j$, is linked to the truncation errors T_j via the matrix equation: $A\underline{e} = \underline{t}$ where A is the tridiagonal matrix in the definition of the finite difference scheme, $\underline{e}$ contains the errors and $\underline{t}$ contains the truncation errors. If each entry in $\underline{t}$ is zero (ie if the truncation error T_j is zero everywhere) then the global error vector $\underline{e} = \underline{0}$ and the global error is zero everywhere.