1. a)

$$u_{tt} - u_{xx} = 0 \text{ with } u(0,t) = u(L,t) = 0,\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x):$$

- Assuming $u = X(x)T(t)$ we can use the PDE with $u_{xx} = X''(x)T(t)$ and $u_{tt} = X(x)T''(t)$ to find $u_{tt} - u_{xx} = XT'' - TX'' = 0$. Dividing by XT (assumed not equal to zero) gives $\frac{T''}{T} = \frac{X''}{X} = \lambda$. Thus we have $X'' - \lambda X = 0$ and $T'' - \lambda T = 0$ with λ constant because $\frac{X''}{X}$ is independent of t and $\frac{T''}{T}$ is independent of x. The homogeneous boundary conditions give $u(0,t) = X(0)T(t) = 0$, $u(L,t) = X(L)T(t) = 0$. Since we seek $XT \neq 0$ this gives $X(0) = X(L) = 0$.
- Solving $X'' - \lambda X = 0$ with $X(0) = X(L) = 0$:

  $\lambda = 0$: we have $X'' = 0$ giving $X = A + Bx$.
  $X(0) = X(L) = 0$ give $A = 0$ and $BL = 0$ so that $A = B = 0$.
  Only the trivial solution arises for $\lambda = 0$.

  $\lambda = \omega^2 > 0$: we have $X'' - \omega^2 X = 0$ giving $X = Ae^{\omega x} + Be^{-\omega x}$. $X(0) = X(L) = 0$ gives $A + B = 0$ and $Ae^{\omega L} + Be^{-\omega L} = 0$,
  i.e. (substituting) $A(e^{\omega L} - e^{-\omega L}) = 0$ so that $A = B = 0$ since $e^{\omega L} - e^{-\omega L} \neq 0$.
  Only trivial solutions arise for $\lambda > 0$.

  $\lambda = -\omega^2 < 0$: we have $X'' + \omega^2 X = 0$ giving $X = A\cos(\omega x) + B\sin(\omega x)$.
  $X(0) = X(L) = 0$ give $A = 0$ and $B\sin(\omega L) = 0$.
  Thus we can have $B \neq 0$ if and only if $\sin(\omega L) = 0$
  or $\omega L = n\pi$ for any $n = 1, 2, 3, \ldots$.

  Hence the only eigenvalues are $\lambda = \lambda_n = -\left(\frac{n\pi}{L}\right)^2$ for $n \in \mathbb{N}$ with corresponding eigenfunctions $X = X_n = \sin(\frac{n\pi x}{L})$.
- Solving $T'' - \lambda T = 0$:
  For any $\lambda = -\left(\frac{n\pi}{L}\right)^2$ we have $T'' + \left(\frac{n\pi}{L}\right)^2 T = 0$. Thus $T = A\cos\left(\frac{n\pi t}{L}\right) + B\sin\left(\frac{n\pi t}{L}\right)$.

- Since the solutions $u = X_n(x)T_n(t)$ all satisfy a homogeneous PDE with homogeneous boundary conditions, the principle of superposition means that any linear combination of such solutions is also a solution. Thus a convergent sum:

$$u = \sum_{n=1}^{\infty} A_n X_n T_n = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

  is also a solution, for constants A_n and B_n.
- At $t = 0$ we have $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ so that:

$$f(x) = \sum_{n=1}^{\infty} A_n X_n T_n(0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} \left(-A_n\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi 0}{L}\right) + B_n\left(\frac{n\pi}{L}\right)\cos\left(\frac{n\pi 0}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right) \\ &= \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

Since the eigenfunctions $X_n = \sin\left(\frac{n\pi x}{L}\right)$ are orthogonal on the interval $[0, L]$ we have: $A_n \int_0^L \sin^2\left(\frac{\pi n x}{L}\right)\, dx = \int_0^L f(x)\, \sin\left(\frac{\pi n x}{L}\right)\, dx$.

Integrating: $\int_0^L \sin^2\left(\frac{\pi n x}{L}\right)\, dx = \int_0^L \frac{1}{2}(1 - \cos\left(\frac{2\pi n x}{L}\right))\, dx = \frac{L}{2} - [\frac{L}{2\pi n}\sin\left(\frac{2\pi n x}{L}\right)]_0^L = \frac{L}{2}$

Thus $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{\pi n x}{L}\right)$, $n = 1, 2, \ldots$. Similarly, for B_n, we obtain the expression, $B_n = \frac{2}{L}\left(\frac{L}{n\pi}\right)\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right) = \frac{2}{n\pi}\int_0^L g(x)\sin\left(\frac{\pi n x}{L}\right) \quad n = 1, 2, \ldots$.

b)

$u_{tt} - u_{xx} = 0$ with $u_x(0,t) = u(\text{L}, t) = 0$, $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$:

- Assuming $u = X(x)T(t)$ we again derive the ODEs: $X'' - \lambda X = 0$ and $T'' - \lambda T = 0$ with λ constant. Here, the homogeneous boundary conditions give $u_x(0,t) = X'(0)T(t) = 0$, $u(L,t) = X(L)T(t) = 0$. Since we seek $XT \neq 0$ this gives $X'(0) = X(L) = 0$.
- Solving $X'' - \lambda X = 0$ with $X'(0) = X(L) = 0$:
  (*as for question 2(b) on exercise sheet 5 — Exercise: repeat for this case*)
  gives eigenvalues $\lambda = -\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$ for $n = 0,\ 1,\ 2,\ \ldots$ and eigenfunctions $X = \cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)$.
- Solving $T'' - \lambda T = 0$:
  For any $\lambda = -\left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2$ we have $T'' + \left(\frac{(n+\frac{1}{2})\pi}{L}\right)^2 T = 0$. Thus:

$$T = A\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi t}{L}\right) + B\sin\left(\frac{\left(n+\frac{1}{2}\right)\pi t}{L}\right).$$

- Since the solutions $u = X_n(x)T_n(t)$ all satisfy a homogeneous PDE with homogeneous boundary conditions, the principle of superposition means that any linear combination of such solutions is also a solution. Thus a convergent sum:

$$u = \sum_{n=0}^{\infty} A_n X_n T_n = \sum_{n=0}^{\infty}\left[A_n\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi t}{L}\right) + B_n\sin\left(\frac{\left(n+\frac{1}{2}\right)\pi t}{L}\right)\right]\cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)$$

  is also a solution, for constants A_n and B_n.
- At $t = 0$ we have $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ so that:

$$f(x) = \sum_{n=1}^{\infty} A_n X_n T_n(0) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)$$

$$g(x) = \sum_{n=0}^{\infty}\left(-A_n\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}\right)\sin\left(\frac{\left(n+\frac{1}{2}\right)\pi 0}{L}\right) + \ldots\right.$$
$$\left. B_n\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}\right)\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi 0}{L}\right)\right)\cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)$$
$$= \sum_{n=0}^{\infty} B_n\left(\frac{\left(n+\frac{1}{2}\right)\pi}{L}\right)\cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right).$$

  Since the eigenfunctions $X_n = \cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)$ are orthogonal under the inner product

$(f,g) = \int_0^L f(x)g(x)\,dx$ we obtain [follow answer to part a)]
$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right), \quad n = 0,1,\dots,$
$B_n = \frac{2}{(n+\frac{1}{2})\pi}\int_0^L g(x)\cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right), \quad n = 0,1,\dots.$

2.

$$u_{xx} + u_{yy} = 0 \text{ with } u(0,y) = u(\pi,y) = 0,\ u(x,0) = 0 \text{ and } u(x,1) = \pi:$$

- Assuming $u = X(x)Y(y)$ we can use the PDE with $u_{xx} = X''(x)Y(y)$ and $u_{yy} = X(x)Y''(y)$ to find $u_{xx} + u_{yy} = X''Y + XY'' = 0$. Dividing by XY (assumed not equal to zero) gives $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$. Thus we have $X'' - \lambda X = 0$ and $Y'' + \lambda Y = 0$ with λ constant because $\frac{X''}{X}$ is independent of y and $\frac{Y''}{Y}$ is independent of x. The homogeneous boundary conditions give $u(0,y) = X(0)Y(y) = 0$, $u(\pi,y) = X(\pi)Y(y) = 0$, $u(x,0) = X(x)Y(0) = 0$. Since we seek $XY \neq 0$ this gives $X(0) = X(\pi) = 0$ and $Y(0) = 0$. The boundary condition $u(x,1) = X(x)Y(1) = \pi$ cannot be used being non-homogeneous.
- Solving $X'' - \lambda X = 0$ with $X(0) = X(\pi) = 0$:
  $\lambda = 0$: we have $X'' = 0$ giving $X = A + Bx$.
  $X(0) = X(\pi) = 0$ give $A = 0$ and $B\pi = 0$ so that $A = B = 0$.
  Only the trivial solution arises for $\lambda = 0$.
  $\lambda = \omega^2 > 0$: we have $X'' - \omega^2 X = 0$ giving $X = Ae^{\omega x} + Be^{-\omega x}$. $X(0) = X(\pi) = 0$ gives $A + B = 0$ and $Ae^{\omega\pi} + Be^{-\omega\pi} = 0$,
  i.e. (substituting) $A(e^{\omega\pi} - e^{-\omega\pi}) = 0$ so that $A = B = 0$ since $e^{\omega\pi} - e^{-\omega\pi} \neq 0$.
  Only trivial solutions arise for $\lambda > 0$.
  $\lambda = -\omega^2 < 0$: we have $X'' + \omega^2 X = 0$ giving $X = A\cos(\omega x) + B\sin(\omega x)$.
  $X(0) = X(\pi) = 0$ give $A = 0$ and $B\sin(\omega\pi) = 0$.
  Thus we can have $B \neq 0$ if and only if $\sin(\omega\pi) = 0$
  or $\omega\pi = n\pi$ for any $n = 1, 2, 3, \dots$.
  Hence the only eigenvalues are $\lambda = \lambda_n = -n^2$ for $n \in \mathbb{N}$
  with corresponding eigenfunctions $X = X_n = \sin(nx)$.
- Solving $Y'' + \lambda Y = 0$ with $Y(0) = 0$:
  For any $\lambda = -n^2$ we have $Y'' - n^2 Y = 0$. Thus $Y = Ae^{ny} + Be^{-ny}$.
  The condition $Y(0) = 0$ gives $A + B = 0$ so, substituting,
  $Y = A(e^{ny} - e^{-ny}) = 2A\frac{e^{ny} - e^{-ny}}{2}$ or $Y = Y_n(y) = \sinh(ny)$, multiplied by any constant.
- Since the solutions $u = X_n(x)Y_n(y)$ all satisfy a homogeneous PDE with homogeneous boundary conditions, the principle of superposition means that any linear combination of such solutions is also a solution. Thus a convergent sum
  $$u = \sum_{n=1}^{\infty} A_n X_n Y_n = \sum_{n=1}^{\infty} A_n \sinh(ny)\sin(nx)$$
  is also a solution, for constants A_n.
- At $y = 1$ we have $u(x,1) = \pi$ so that $\sum_{n=1}^{\infty} A_n \sinh(n)\sin(nx) = \pi$.
  Since the eigenfunctions $X_n = \sin(nx)$ are orthogonal under the inner product $(f,g) = \int_0^\pi f(x)g(x)\,dx$ we have $A_n \sinh(n)\int_0^\pi \sin^2(nx)\,dx = \pi\int_0^\pi \sin(nx)\,dx$.

  Integrating: $\int_0^\pi \sin^2(nx)\,dx = \int_0^\pi \frac{1}{2}(1 - \cos(2nx))\,dx = \frac{\pi}{2} - [\frac{1}{2n}\sin(2nx)]_0^\pi = \frac{\pi}{2}$
  and $\int_0^\pi \sin(nx)\,dx = [-\frac{1}{n}\cos(nx)]_0^\pi = \frac{2}{n}$ if n is odd or 0 if n is even.
  Thus $A_n \sinh(n)\frac{\pi}{2} = \pi\frac{2}{n}$ or 0 so that $A_n = \frac{4}{n\sinh(n)}$ or 0. Setting $n = 2k+1$, the solution can therefore be written as
  $$u = \sum_{k=0}^{\infty} \frac{4}{2k+1}\frac{\sinh\big((2k+1)y\big)}{\sinh(2k+1)}\sin\big((2k+1)x\big).$$

3. a) In polar co-ordinates the Laplacian is $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$ and so the wave equation on a disk (with $c^2 = 1$) in polar co-ordinates is:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

If the solution u does not depend on the angle θ and only depends on the distance r from the centre of the drum then $u_{\theta\theta} = 0$ and the reduced equation for $u = u(r,t)$ is

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}. \qquad (1)$$

b) Substituting $u(r,t) = R(r)T(t)$ into the PDE (1) gives: $RT'' = TR'' + \frac{1}{r}R'T$. Rearranging and setting equal to the separation constant gives $\frac{T''}{T} = \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = \lambda$. This decouples into two ODEs:

$$T'' - \lambda T = 0, \qquad r^2R'' + rR' - \lambda r^2 R = 0.$$

(The equation in R has been multiplied by r^2 on both sides.) Substituting $u(r,t) = R(r)T(t)$ into the boundary conditions at $r = 0$ and $r = a$ gives:

$$u(a,t) = 0 \Rightarrow R(a) = 0, \qquad |u(0,t)| < \infty \Rightarrow |R(0)| < \infty.$$

We are told that we are only interested in negative eigenvalues. So write $\lambda = -\omega^2$. The ODE for $R(r)$ is then

$$r^2R'' + rR' + \omega^2 r^2 R = 0.$$

This is Bessel's equation of order zero (the variable is now r instead of x). So the general form of the solution is:

$$R(r) = c_1 J_0(\omega r) + c_2 Y_0(\omega r)$$

where c_1 and c_2 are arbitrary constants. Note that $Y_0(0) = -\infty$ (ask MATLAB to evaluate Y_0 at 0) and $J_0(0)$ is bounded. To satisfy the condition $|R(0)| < \infty$ we therefore need $c_2 = 0$. To satisfy the condition at $r = a$ we need $0 = R(a) = c_1 J_0(\omega a)$. Hence, non-zero solutions $R(r)$ exist when $J_0(\omega a) = 0$ i.e. when ωa is a root of J_0 or

$$\omega = \omega_n = \frac{\alpha_n}{a}, \qquad n = 1, 2, 3, \ldots$$

where α_n denotes a root of J_0. The corresponding eigenfunctions are $R_n(r) = c_n J_0(\omega_n r/a)$. Next, given that we know $\lambda_n = -\omega_n^2$ we can solve the ODE for $T(t)$ for each value of λ_n. We have the general solution

$$T_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t).$$

Hence, the separated solutions corresponding to negative values of λ are

$$u_n(r,t) = (A_n \cos(\omega_n t) + B_n \sin(\omega_n t))\, J_0(\omega_n r/a), \qquad n = 1, 2, 3 \ldots$$

c) If $a = 1$ the separated solutions are

$$u_n(r,t) = (A_n \cos(\alpha_n t) + B_n \sin(\alpha_n t))\, J_0(\alpha_n r), \qquad n = 1, 2, 3 \ldots$$

We have

$$u_n(r,0) = A_n J_0(\alpha_n r), \qquad n = 1, 2, 3 \ldots$$

and

$$\frac{\partial u_n(r,0)}{\partial t} = B_n \alpha_n J_0(\alpha_n r), \qquad n = 1, 2, 3 \ldots$$

The initial conditions are: $u(r,0) = J_0(\alpha_3 r)$, $\frac{\partial u}{\partial t}(r,0) = 0$. The individual solution $(n = 3)$

$$u_3(r,t) = (A_3 \cos(\alpha_3 t) + B_3 \sin(\alpha_3 t))\, J_0(\alpha_3 r),$$

satisfies these initial conditions if we set $A_3 = 1$ and $B_3 = 0$. (Note we don't have to add all the solutions together to get a solution which satisfies the initial conditions if the initial conditions look like one of the eigenfunctions). Our final solution then is

$$u(r,t) = \cos(\alpha_3 t) J_0(\alpha_3 r).$$