1. Which of the following PDEs can be solved using the method of separation of variables? (Do not solve them.)
   a) No - coefficient of u_{xx} depends on x and t (impossible to rearrange)
   b) Yes
   c) Yes
   d) Yes
   e) Yes
   f) No (mixed derivative term)

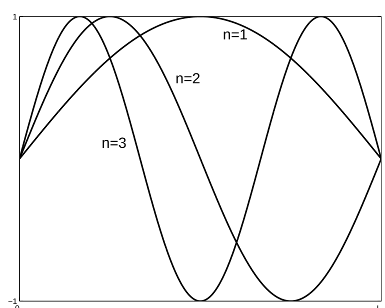
2. (a) $X'' - \lambda X = 0$ with $X(0) = X(l) = 0$. Taking $X = X(x)$:

   $\lambda = 0$ gives $X'' = 0$, so $X = a + bx$. BCs give $X(0) = a = 0$, so $a = 0$, and $X(l) = bl = 0$, so $b = 0$. Hence $X \equiv 0$.

   $\lambda > 0$, with $\lambda = \omega^2 \neq 0$, gives $X'' - \omega^2 X = 0$, so $X = ae^{\omega x} + be^{-\omega x}$.
   BCs give $X(0) = a + b = 0$ and $X(l) = ae^{\omega l} + be^{-\omega l} = 0$, so $b = -a$ and $a(e^{\omega l} - e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} - e^{-\omega l} \neq 0$). Hence $X \equiv 0$.

   $\lambda < 0$, with $\lambda = -\omega^2 \neq 0$, gives $X'' + \omega^2 X = 0$, so $X = a\cos(\omega x) + b\sin(\omega x)$.
   BCs give $X(0) = a\cos 0 + b\sin 0 = a = 0$, so $a = 0$, and $X(l) = b\sin(\omega l) = 0$, so it is possible to have $b \neq 0$ only if $\sin(\omega l) = 0$.
   That is, if $\omega l = n\pi$ for $n = 1, 2, 3, \ldots$ then $X = b\sin(n\pi x/l)$.

   Hence we find the eigenvalues, $\lambda = -(n\pi/l)^2$ for $n = 1, 2, 3, \ldots$.
   and the corresponding eigenfunctions, $X = \sin(n\pi x/l)$.



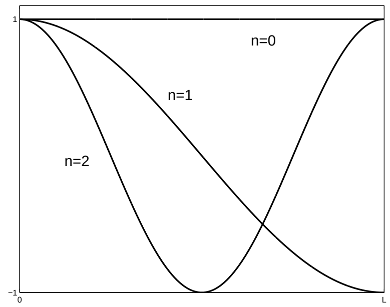
   (b) $Y'' - \lambda Y = 0$ with $Y'(0) = Y'(l) = 0$. Taking $Y = Y(y)$:

   $\lambda = 0$ gives $Y'' = 0$, so $Y = a + by$. BCs give $Y'(0) = b = 0$, so $b = 0$, and $Y'(l) = b = 0$, so $b = 0$ (again). Thus any value of $a \neq 0$ is admissible and so $Y = a$ is a solution if $\lambda = 0$.

   $\lambda > 0$, with $\lambda = \omega^2 \neq 0$, gives $Y'' - \omega^2 Y = 0$, so $Y = ae^{\omega y} + be^{-\omega y}$.
   BCs give $Y'(0) = a\omega - b\omega = 0$ and $Y'(l) = a\omega e^{\omega l} - b\omega e^{-\omega l} = 0$, so $b = a$ and $a\omega(e^{\omega l} - e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} - e^{-\omega l} \neq 0$). Hence $Y \equiv 0$.

   $\lambda < 0$, with $\lambda = -\omega^2 \neq 0$, gives $Y'' + \omega^2 Y = 0$, so $Y = a\cos(\omega y) + b\sin(\omega y)$.
   BCs give $Y'(0) = -a\omega\sin 0 + b\omega\cos 0 = b\omega = 0$, so $b = 0$, and
   $Y'(l) = -a\omega\sin(\omega l) = 0$, so it is possible to have $a \neq 0$ only if $\sin(\omega l) = 0$.
   That is, if $\omega l = n\pi$ for $n = 1, 2, 3, \ldots$ then $Y = a\cos(n\pi y/l)$.

   Hence we find the eigenvalues, $\lambda = -(n\pi/l)^2$ for $n = 0, 1, 2, \ldots$ and corresponding eigenfunctions, $Y = \cos(n\pi y/l)$. (Note: this is the constant 1 for $n = 0$).



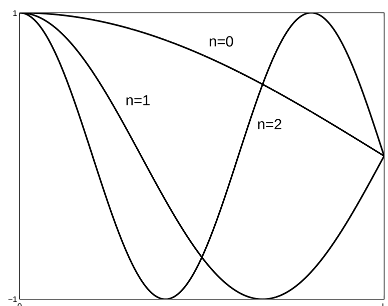
(c) $Z'' - \lambda Z = 0$ with $Z'(0) = Z(l) = 0$. Taking $Z = Z(z)$:

$\lambda = 0$ gives $Z'' = 0$, so $Z = a + bz$. BCs give $Z'(0) = b = 0$, so $b = 0$, and $Z(l) = a = 0$, so $a = 0$. Hence $Z \equiv 0$.

$\lambda > 0$, with $\lambda = \omega^2 \neq 0$, gives $Z'' - \omega^2 Z = 0$, so $Z = ae^{\omega z} + be^{-\omega z}$.
BCs give $Z'(0) = a\omega - b\omega = 0$ and $Z(l) = ae^{\omega l} + be^{-\omega l} = 0$, so $b = a$ and $a(e^{\omega l} + e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} + e^{-\omega l} \neq 0$). Hence $Z \equiv 0$.

$\lambda < 0$, with $\lambda = -\omega^2 \neq 0$, gives $Z'' + \omega^2 Z = 0$, so $Z = a\cos(\omega z) + b\sin(\omega z)$.
BCs give $Z'(0) = -a\omega\sin 0 + b\omega\cos 0 = b\omega = 0$, so $b = 0$,
and $Z(l) = a\cos(\omega l) = 0$, so it is possible to have $a \neq 0$ only if $\cos(\omega l) = 0$.
That is, if $\omega l = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2, \ldots$ then $Z = a\cos\big((n + \frac{1}{2})\pi z/l\big)$.

Hence we find the eigenvalues, $\lambda = -\big((n + \frac{1}{2})\pi/l\big)^2$ for $n = 0, 1, 2, \ldots$
and the corresponding eigenfunctions, $Z = \cos\big((n + \frac{1}{2})\pi z/l\big)$.



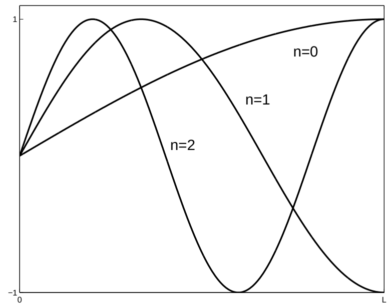
(d) $F'' - \lambda F = 0$ with $F(0) = F'(l) = 0$. Taking $F = F(f)$:

$\lambda = 0$ gives $F'' = 0$, so $F = a + bf$. BCs give $F(0) = a = 0$, so $a = 0$, and $F'(l) = b = 0$, so $b = 0$. Hence $F \equiv 0$.

$\lambda > 0$, with $\lambda = \omega^2 \neq 0$, gives $F'' - \omega^2 F = 0$, so $F = ae^{\omega f} + be^{-\omega f}$.
BCs give $F(0) = a + b = 0$ and $F'(l) = a\omega e^{\omega l} - b\omega e^{-\omega l} = 0$, so $b = -a$ and $a\omega(e^{\omega l} + e^{-\omega l}) = 0$ so that $a = b = 0$ (since $e^{\omega l} + e^{-\omega l} \neq 0$). Hence $F \equiv 0$.

$\lambda < 0$, with $\lambda = -\omega^2 \neq 0$, gives $F'' + \omega^2 F = 0$, so $F = a\cos(\omega f) + b\sin(\omega f)$.
BCs give $F(0) = a\cos 0 + b\sin 0 = a = 0$, so $a = 0$, and
$F'(l) = b\omega\cos(\omega l) = 0$, so it is possible to have $b \neq 0$ only if $\cos(\omega l) = 0$.
That is, if $\omega l = (n + \frac{1}{2})\pi$ for $n = 0, 1, 2, \ldots$ then $F = b\sin\big((n + \frac{1}{2})\pi f/l\big)$.

Hence we find the eigenvalues, $\lambda = -\big((n + \frac{1}{2})\pi/l\big)^2$ for $n = 0, 1, 2, \ldots$
and the corresponding eigenfunctions $F = \sin\big((n + \frac{1}{2})\pi f/l\big)$.



3. (a) $1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ leads to

$$\int_0^l 1 \sin \frac{m\pi x}{l}\, dx = \sum_{n=1}^{\infty} b_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = b_m \int_0^l \sin^2 \frac{m\pi x}{l}\, dx$$

(due to orthogonality). Evaluating:

$$\int_0^l \sin \frac{m\pi x}{l}\, dx = -\frac{l}{m\pi}\left[\cos \frac{m\pi x}{l}\right]_0^l = -\frac{l}{m\pi}\big((-1)^m - 1\big) = \begin{cases} \frac{2l}{m\pi} & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even.} \end{cases}$$

$$\int_0^l \sin^2 \frac{m\pi x}{l}\, dx = \int_0^l \tfrac{1}{2}\left(1 - \cos \frac{2m\pi x}{l}\right) dx = \tfrac{1}{2}l - \tfrac{1}{2}\frac{l}{2m\pi}\left[\sin \frac{2m\pi x}{l}\right]_0^l = \tfrac{1}{2}l.$$

Hence $\frac{2l}{(2k+1)\pi} = b_{2k+1} \times \frac{1}{2}l$ or $b_{2k+1} = \frac{4}{(2k+1)\pi}$ with $b_{2k} = 0$

and so $1 = \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi} \sin \frac{(2k+1)\pi x}{l}.$

(b) $x = \sum_{n=0}^{\infty} a_n \cos \frac{n\pi x}{l}$ leads to

$$\int_0^l x \cos \frac{m\pi x}{l}\, dx = \sum_{n=0}^{\infty} a_n \int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = a_m \int_0^l \cos^2 \frac{m\pi x}{l}\, dx$$

(due to orthogonality). Evaluating:

For $m = 0$: $\int_0^l x \cos \frac{m\pi x}{l}\, dx = \int_0^l x\, dx = \left[\frac{1}{2}x^2\right]_0^l = \frac{1}{2}l^2$

and $\int_0^l \cos^2 \frac{m\pi x}{l}\, dx = \int_0^l dx = l$ so that $\frac{1}{2}l^2 = a_0 \times l$ giving $a_0 = \frac{1}{2}l.$

Otherwise, for $m > 0$:

$$\int_0^l x \cos \frac{m\pi x}{l}\, dx = \left[x\frac{l}{m\pi} \sin \frac{m\pi x}{l}\right]_0^l - \int_0^l \frac{l}{m\pi} \sin \frac{m\pi x}{l}\, dx = \left(\frac{l}{m\pi}\right)^2 \left[\cos \frac{m\pi x}{l}\right]_0^l$$

$$= \left(\frac{l}{m\pi}\right)^2 \big((-1)^m - 1\big) = \begin{cases} -2\left(\frac{l}{m\pi}\right)^2 & \text{for } m \text{ odd} \\ 0 & \text{for } m \text{ even.} \end{cases}$$

$$\int_0^l \cos^2 \frac{m\pi x}{l}\, dx = \int_0^l \tfrac{1}{2}\left(1 + \cos \frac{2m\pi x}{l}\right) dx = \tfrac{1}{2}l + \tfrac{1}{2}\frac{l}{2m\pi}\left[\sin \frac{2m\pi x}{l}\right]_0^l = \tfrac{1}{2}l.$$

Hence $-\frac{2l^2}{(2k+1)^2\pi^2} = a_{2k+1} \times \frac{1}{2}l$ or $a_{2k+1} = -\frac{4l}{(2k+1)^2\pi^2}$ with $a_{2k} = 0$

and so $x = \frac{1}{2}l - \sum_{k=0}^{\infty} \frac{4l}{(2k+1)^2\pi^2} \cos \frac{(2k+1)\pi x}{l}.$

(c) $\pi = \sum_{n=0}^{\infty} a_n \cos \frac{(n+\frac{1}{2})\pi x}{l}$ leads to

$$\int_0^l \pi\cos\frac{(m+\frac12)\pi x}{l}\,dx = \sum_{n=0}^{\infty} a_n \int_0^l \cos\frac{(n+\frac12)\pi x}{l}\cos\frac{(m+\frac12)\pi x}{l}\,dx$$
$$= a_m \int_0^l \cos^2\frac{(m+\frac12)\pi x}{l}\,dx$$

(due to orthogonality).

Evaluating:

$$\int_0^l \pi\cos\frac{(m+\frac12)\pi x}{l}\,dx = \pi\frac{l}{(m+\frac12)\pi}\left[\sin\frac{(m+\frac12)\pi x}{l}\right]_0^l = \frac{(-1)^m\, l}{m+\frac12}.$$

$$\int_0^l \cos^2\frac{(m+\frac12)\pi x}{l}\,dx = \int_0^l \tfrac12\left(1+\cos\frac{(2m+1)\pi x}{l}\right)dx$$
$$= \tfrac12 l + \tfrac12\frac{l}{(2m+1)\pi}\left[\sin\frac{(2m+1)\pi x}{l}\right]_0^l = \tfrac12 l.$$

Hence $\dfrac{(-1)^n\, l}{n+\frac12} = a_n \times \frac12 l$ or $a_n = \dfrac{2(-1)^n}{n+\frac12}$

and so $\pi = \displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^n}{n+\frac12}\cos\frac{(n+\frac12)\pi x}{l}.$

(d) $x = \displaystyle\sum_{n=0}^{\infty} b_n \sin\frac{(n+\frac12)\pi x}{l}$ leads to

$$\int_0^l x\sin\frac{(m+\frac12)\pi x}{l}\,dx = \sum_{n=0}^{\infty} b_n \int_0^l \sin\frac{(n+\frac12)\pi x}{l}\sin\frac{(m+\frac12)\pi x}{l}\,dx = b_m\int_0^l \sin^2\frac{(m+\frac12)\pi x}{l}\,dx$$

(through orthogonality). Evaluating:

$$\int_0^l x\sin\frac{(m+\frac12)\pi x}{l}\,dx = \left[-x\frac{l}{(m+\frac12)\pi}\cos\frac{(m+\frac12)\pi x}{l}\right]_0^l + \int_0^l \frac{l}{(m+\frac12)\pi}\cos\frac{(m+\frac12)\pi x}{l}\,dx$$
$$= \left[\left(\frac{l}{(m+\frac12)\pi}\right)^2 \sin\frac{(m+\frac12)\pi x}{l}\right]_0^l = \frac{(-1)^m\, l^2}{(m+\frac12)^2\pi^2}.$$

$$\int_0^l \sin^2\frac{(m+\frac12)\pi x}{l}\,dx = \int_0^l \tfrac12\left(1-\cos\frac{(2m+1)\pi x}{l}\right)dx$$
$$= \tfrac12 l - \tfrac12\frac{l}{(2m+1)\pi}\left[\sin\frac{(2m+1)\pi x}{l}\right]_0^l = \tfrac12 l.$$

Hence $\dfrac{(-1)^m\, l^2}{(n+\frac12)^2\pi^2} = b_n \times \frac12 l$ or $b_n = \dfrac{2(-1)^m\, l}{(n+\frac12)^2\pi^2}$

and so $x = \displaystyle\sum_{n=0}^{\infty} \frac{2(-1)^m\, l}{(n+\frac12)^2\pi^2}\sin\frac{(n+\frac12)\pi x}{l}.$

4. We have: $u_t = u_{xx}$ with the boundary conditions $u_x(0,t) = 0$ and $u(a,t) = 0$.

Assuming $u = T(t)X(x)$ and that $T(t) \not\equiv 0$ gives $T'X = TX''$ or $\dfrac{X''}{X} = \dfrac{T'}{T} = \lambda$ with the boundary conditions giving $T(t)X'(0) = 0$ and $T(t)X(a) = 0$. So $T' - \lambda T = 0$ and $X'' - \lambda X = 0$ with boundary conditions $X'(0) = X(a) = 0$.

You should recognise the boundary value problem for $X(x)$ as being the same one considered in question 2 c) (with ℓ replaced by a). Hence, the eigenfunctions are $X_n(x) = c_n\cos((\frac12+n)\pi x/a)$ (for arbitrary constants c_n) and the corresponding eigenvalues are $\lambda = -((\frac12+n)\pi/a)^2$, $n = 0, 1, 2, \ldots,$

Also, for a fixed λ we have $T = \text{const.} \times e^{\lambda t}$. The separated solutions are therefore

$$u = u_n(x,t) = A_n e^{-\left((\frac12+n)\pi/a\right)^2 t}\cos\left(\left(\frac12+n\right)\pi x/a\right), \qquad n = 0, 1, 2, \ldots$$

where A_n is an arbitrary constant. These are all linearly independent. Applying the principle of superposition, an infinite series solution is

$$u(x,t) = \sum_{n=0}^{\infty} A_n e^{-((\frac{1}{2}+n)\pi/a)^2 t} \cos((\tfrac{1}{2}+n)\pi x/a).$$

To fit the initial condition at $t = 0$, we now need to choose A_n such that

$$a = \sum_{n=0}^{\infty} A_n \cos((\tfrac{1}{2}+n)\pi x/a).$$

Notice that this was done in question 3c) (for $a = \pi$). All of the eigenfunctions X_n are orthogonal on $[0, a]$ so that

$$\int_0^a a \cos\left((\tfrac{1}{2}+n)\pi x/a\right)\, dx = A_n \int_0^a \cos^2\left((\tfrac{1}{2}+n)\pi x/a\right)\, dx.$$

Evaluating: $\displaystyle\int_0^a a \cos\left((\tfrac{1}{2}+n)\pi x/a\right)\, dx = a\left[\frac{a}{(\frac{1}{2}+n)\pi} \sin\left((\tfrac{1}{2}+n)\pi x/a\right)\right]_0^a = \frac{a^2\,(-1)^n}{(\frac{1}{2}+n)\pi}$

and $$\int_0^a \cos^2\left((\tfrac{1}{2}+n)\pi x/a\right)\, dx = \int_0^a \tfrac{1}{2}\left\{1 + \cos\left(2(\tfrac{1}{2}+n)\pi x/a\right)\right\}\, dx$$
$$= \tfrac{1}{2}\left[x + \frac{a}{(1+2n)\pi} \sin\left((1+2n)\pi x/a\right)\right]_0^a = \tfrac{1}{2}a$$

Hence,

$$A_n = \frac{4a\,(-1)^n}{(1+2n)\pi}$$

and the full series solution becomes

$$u(x,t) = \sum_{n=0}^{\infty} \frac{4a\,(-1)^n}{(1+2n)\pi}\, e^{-\left((\frac{1}{2}+n)\pi/a\right)^2 t} \cos\left((\tfrac{1}{2}+n)\pi x/a\right).$$