1. (Easy question) Differentiating $u(x,y) = x^2y^4 + e^{xy}$ gives,

$$u_x = 2xy^4 + ye^{xy}, \quad u_y = 4x^2y^3 + xe^{xy}.$$

$$u_{xx} = 2y^4 + y^2e^{xy} \quad u_{yy} = 12x^2y^2 + x^2e^{xy}, \quad u_{xy} = 8xy^3 + xye^{xy} + e^{xy} = u_{yx}.$$

2. Differentiation $u(x,y) = \ln\sqrt{x^2+y^2}$ gives

$$u_x = x\left(x^2+y^2\right)^{-1}, \quad u_y = y\left(x^2+y^2\right)^{-1},$$

$$u_{xx} = -2x^2\left(x^2+y^2\right)^{-2} + \left(x^2+y^2\right)^{-1} \quad u_{yy} = -2y^2\left(x^2+y^2\right)^{-2} + \left(x^2+y^2\right)^{-1}$$

Adding the two second derivatives gives:

$$u_{xx}+u_{yy} = -2\left(x^2+y^2\right)\left(x^2+y^2\right)^{-2}+2\left(x^2+y^2\right)^{-1} = -2\left(x^2+y^2\right)^{-1}+2\left(x^2+y^2\right)^{-1} = 0.$$

Hence the given $u(x,y)$ satisfies Laplace's equation in Cartesian co-ordinates. Converting to polar co-ordinates, using the relations $x = r\cos\theta$, $y = r\sin\theta$ gives $u(x,y) = u(x(r,\theta), y(r,\theta)) = \ln r$. From the handout on classical PDEs, Laplace's equation in polar co-ordinates is $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. Differentiating u gives

$$u_r = \frac{1}{r}, u_{rr} = -\frac{1}{r^2}, u_\theta = 0 = u_{\theta,\theta},$$

and so we obtain

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = -\frac{1}{r^2} + \frac{1}{r}\frac{1}{r} + 0 = 0.$$

3. Differentiating $u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}$, with respect to x gives,

$$u_x = -\frac{x}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{4t}}, \quad u_{xx} = -\frac{1}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{4t}}\left(1 - \frac{x^2}{2t}\right).$$

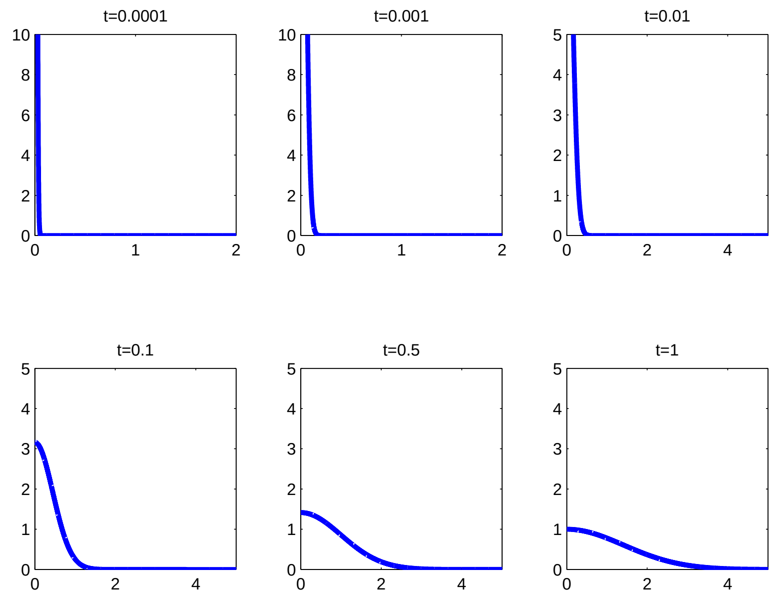
Differentiating $u(x,t) = \frac{e^{-\frac{x^2}{4t}}}{\sqrt{t}}$, with respect to t gives,

$$u_t = \frac{x^2}{4}t^{-\frac{5}{2}}e^{-\frac{x^2}{4t}} - \frac{1}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{4t}} = -\frac{1}{2}t^{-\frac{3}{2}}e^{-\frac{x^2}{4t}}\left(1 - \frac{x^2}{2t}\right).$$

Hence, the suggested $u(x,t)$ does satisfy the equation: $u_t = u_{xx}$. You can also check that $u(x,t)$ satisfies the following conditions:

$$\begin{array}{lll} u(0,t) = \frac{1}{\sqrt{t}} & u(\infty,t) = 0 & u_x(0,t) = 0, \\ u(0,0) = \infty & u_t(x,0) = -\infty & \end{array}$$

We can consider $u(x,t)$ as representing the temperature in a one-dimensional wire of infinite length. At time $t = 0$ there is a sudden influx of heat at $x = 0$. This heat then diffuses along the bar from the point $x = 0$. Since $u_x(0,t) = 0$, the end of the wire at $x = 0$ is 'insulated' and no heat flows out of the bar in the left-hand direction. See below for graphs of $u(x,t)$ for various values of t. In each picture, the horizontal axis is the x axis and we plot only a section of it (since the wire has infinite length).



4. The one-dimensional wave equation can be written in operator notation as $Lu = 0$ where $Lu = u_{tt} - c^2 u_{xx}$. Now, give any two solutions u_1 and u_2 (and one is not a scalar multiple of the other) we need to show that i) $L(u_1 + u_2) = Lu_1 + Lu_2$ and ii) $L(\alpha u_1) = \alpha L u_1$ for any constant α. Both of these conditions are trivial to check:

$$\begin{aligned} L(u_1 + u_2) &= \frac{\partial^2}{\partial t^2}(u_1 + u_2) - c^2 \frac{\partial^2}{\partial x^2}(u_1 + u_2) \\ &= \frac{\partial^2 u_1}{\partial t^2} + \frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} \\ &= \left(\frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} \right) + \left(\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} \right) = Lu_1 + Lu_2. \\ L(\alpha u_1) &= \frac{\partial^2 \alpha u_1}{\partial t^2} - c^2 \frac{\partial^2 \alpha u_1}{\partial x^2} = \alpha \left(\frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} \right) = \alpha L u_1. \end{aligned}$$

Similarly, the two-dimensional version of Laplace's equation is $Lu = u_{xx} + u_{yy} = 0$. We have,

$$\begin{aligned} L(u_1 + u_2) &= \frac{\partial^2}{\partial x^2}(u_1 + u_2) + \frac{\partial^2}{\partial y^2}(u_1 + u_2) \\ &= \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 u_2}{\partial y^2} \\ &= \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) + \left(\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_2}{\partial y^2} \right) = Lu_1 + Lu_2. \\ L(\alpha u_1) &= \frac{\partial^2 \alpha u_1}{\partial x^2} + \frac{\partial^2 \alpha u_1}{\partial y^2} = \alpha \left(\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} \right) = \alpha L u_1. \end{aligned}$$

5. i) First-order. Non-linear (due to the term $5uu_y$).
   ii) Second-order. Linear. Homogeneous.
   iii) First-order. Linear. Homogeneous.
   iv) Second-order. Linear. Non-homogeneous (due to the term $2xy$).

v) Fourth-order. Linear. Homogeneous.

vi) Second-order. Non-linear (due to the term $\sinh u$).

6. Classify the following PDEs as being either elliptic, parabolic or hyperbolic:

   i) Linear and hyperbolic. $B^2 - 4AC = +4c^2 > 0$. (Parabolic if $c = 0$.)

   ii) Linear and elliptic. $B^2 - 4AC = -4 < 0$.

   iii) Linear and hyperbolic. $B^2 - 4AC = +4 > 0$.

   iv) Linear. $B^2 - 4AC = -4x$. Hence the PDE is elliptic when $x > 0$, parabolic when $x = 0$ and hyperbolic when $x < 0$.

   v) Linear. $B^2 - 4AC = 4\left(y^2 - x\right)$. Hence, the equation is elliptic when $y^2 < x$, parabolic when $y^2 = x$ and hyperbolic when $y^2 > x$.

7. i) Find $u(x,t)$ satisfying the one-dimensional heat equation:

$$\begin{array}{lll} & u_t = 3.2u_{xx}, & \text{for } 0 < x < \pi, t > 0, \\ \text{such that} & u_x(0,t) = 0, u_x(\pi,t) = 0, & u(x,0) = \cos(2x). \end{array}$$

   ii) Find $u(x,t)$ satisfying the one-dimensional wave equation:

$$\begin{array}{lll} & u_{tt} = (1.5)^2 u_{xx}, & \text{for } 0 < x < 2, 0 < t < 2, \\ \text{such that} & u(0,t) = 0, u(2,t) = 0, & \\ \text{and} & u(x,0) = \sin(3x), u_t(x,0) = 0 & \end{array}$$

   Note that *two* initial conditions are required to define a unique solution to the wave equation.

   iii) Find $u(x,t)$ satisfying the two-dimensional version of Laplace's equation:

$$\begin{array}{lll} & u_{xx} + u_{yy} = 0, & \text{for } 0 < x < 10, 0 < y < 10, \\ \text{such that} & u_y(x,0) = 0, u_y(x,10) = 0, & u(0,y) = 5, u(10,y) = 0. \end{array}$$