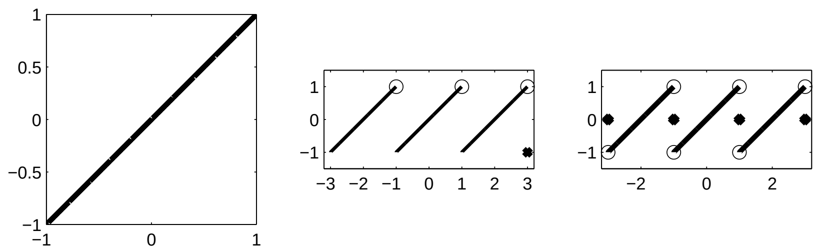
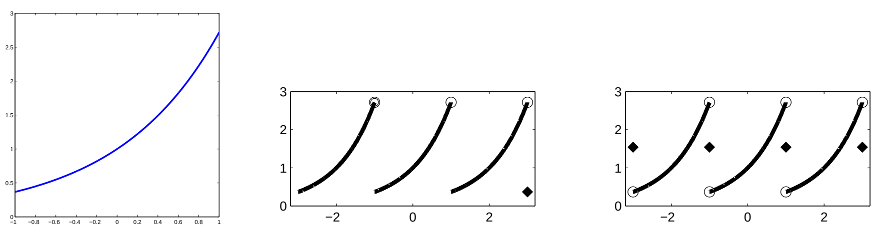
1. For each of the given functions we plot below, $f(x)$ (left), the periodic extension $\tilde{f}(x)$ (middle) and the Fourier Series (right).

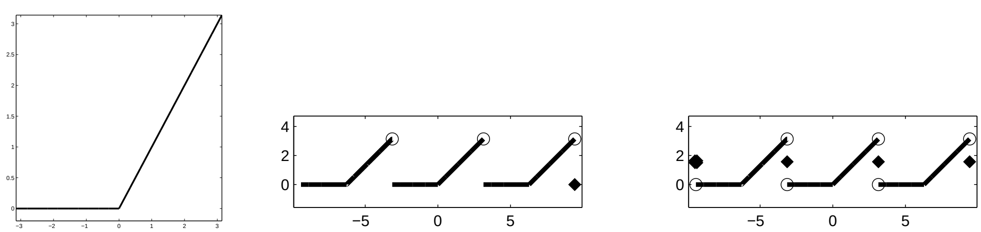
   a) $f(x) = x$ on $[-1, 1]$.



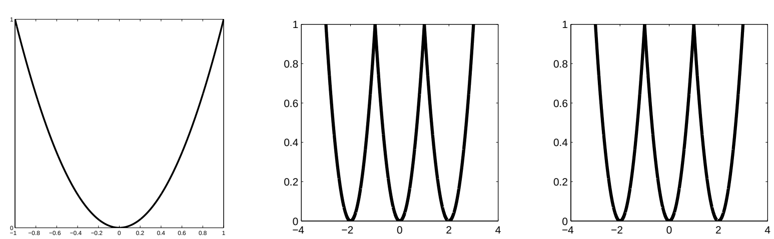
   b) $f(x) = \exp(x)$ on $[-1, 1]$.



   c) $f(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ x & 0 < x \leq \pi \end{cases}$



   d) $f(x) = x^2, x \in [-1, 1]$



2. a) It should be obvious that the function $\sin\left(\frac{\pi x}{L}\right)$ is already in the form of a Fourier series with coefficients $a_0 = a_n = 0, n = 1, 2, \ldots$ and $b_n = 0, n \neq 1$ and $b_1 = 1$. Applying the definition, we have

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi x}{L}\right),$$

and working out the coefficients rigourously, we obtain,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} \sin\left(\frac{\pi x}{L}\right) dx = \frac{1}{2L}\left[\frac{L}{\pi}\cos\left(\frac{\pi x}{L}\right)\right]_{-L}^{L} = \frac{1}{2L}\left[\frac{L}{\pi}(-1+1)\right] = 0,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad \text{(using question 4 on sheet 2)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad \text{unless } n = 1 \text{ (using question 4 on sheet 2)}$$

and finally,

$$b_1 = \frac{1}{L}\int_{-L}^{L} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{1}{2L}\int_{-L}^{L} 1 - \cos\left(\frac{2\pi x}{L}\right) dx = \frac{1}{2L}\left[x - \frac{L}{2\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_{-L}^{L} = 1$$

(as we suspected).

b) Using integration by parts, where necessary, we have

$$a_0 = \frac{1}{2L}\int_{-L}^{L} x\,dx = \frac{1}{2L}\left[\frac{x^2}{2}\right]_{-L}^{L} = 0,$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\left[\left\{\frac{xL}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right\}_{-L}^{L} - \int_{-L}^{L}\left(\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)\right] dx \\ &= \frac{1}{L}\left[0 - \frac{L^2}{n^2\pi^2}\left[-\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{L}\right] = \frac{-L}{n\pi}\left[-\cos(n\pi) + \cos(-n\pi)\right] = 0. \end{aligned}$$

(Alternatively - just note that x is an odd function) and the a_n coefficients are always zero for odd functions. (See lecture notes for explanation).

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\left[\left\{-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right\}_{-L}^{L} + \int_{-L}^{L}\left(\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right)\right] dx \\ &= \frac{1}{L}\left[-\frac{L^2}{n\pi}\left(\cos(n\pi) + \cos(-n\pi)\right)\right] + \frac{L}{n\pi}\left[\sin\left(\frac{n\pi x}{L}\right)\frac{L}{n\pi}\right]_{-L}^{L} = \frac{-2L}{n\pi}\cos(n\pi) + 0 \\ &= \frac{-2L}{n\pi}(-1)^n = \frac{2L}{n\pi}(-1)^{n+1}. \end{aligned}$$

Hence, the Fourier series associated with the periodic extension of $f(x)$ is

$$\sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right).$$

c) Using integration by parts, where necessary, we have

$$a_0 = \frac{1}{2L}\int_{-L}^{0} 1\,dx + \frac{1}{2L}\int_{0}^{L} 2\,dx = \frac{1}{2} + 1 = \frac{3}{2}.$$

$$
\begin{aligned}
a_n &= \frac{1}{L}\left(\int_{-L}^{0} \cos\left(\frac{n\pi x}{L}\right) + \int_{0}^{L} 2\cos\left(\frac{n\pi x}{L}\right) dx\right) \\
&= \frac{1}{L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{-L}^{0} + \frac{2}{L}\left[\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_{0}^{L} = 0 + 0 = 0.
\end{aligned}
$$

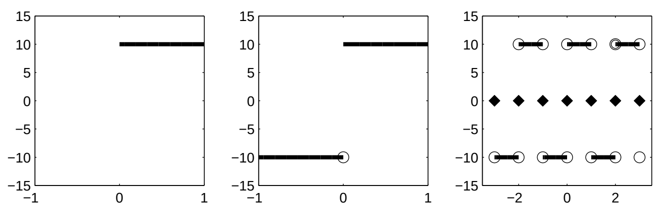
$$
\begin{aligned}
b_n &= \frac{1}{L}\left(\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right) + \int_{0}^{L} 2\sin\left(\frac{n\pi x}{L}\right) dx\right) \\
&= \frac{1}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{-L}^{0} + \frac{2}{L}\left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_{0}^{L} \\
&= \frac{-1}{n\pi} + \frac{1}{n\pi}\cos(n\pi) - \frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi} = \frac{1}{n\pi}\left[1 + (-1)^{n+1}\right].
\end{aligned}
$$

Hence, the Fourier series assocaited with the periodic extension of $f(x)$ is

$$
\frac{3}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left[1 + (-1)^{n+1}\right] \sin\left(\frac{n\pi x}{L}\right).
$$

3. a) $f(x) = |x|$ is an even function since $f(-x) = |-x| = |x| = f(x)$. b) $f(x) = x^{175}$ is an odd function since $f(-x) = (-x)^{175} = (-1)^{175}x^{175} = -(x^{175}) = -f(x)$. Note that any odd power of x is an odd function. c) $f(x) = x^{-1}$ is an odd function since $f(-x) = (-x)^{-1} = (-1)^{-1}x^{-1} = -(x^{-1}) = -f(x)$. d) $f(x) = \cos\left(\frac{4\pi x}{3}\right)$ is an even function since $f(-x) = \cos\left(-\frac{4\pi x}{3}\right) = \cos\left(\frac{4\pi x}{3}\right) = f(x)$.

4. a) Below we plot $f(x) = 10$, on the interval $[0, 1]$, (left), the odd extension $f_{odd}(x)$ on the interval $[-1, 1]$ (middle), and the Fourier sine series associated with the periodic extension $\tilde{f}_{odd}(x)$ (right).



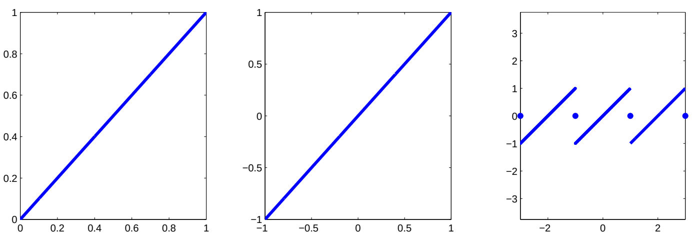
To compute the coefficients of the Fourier sine series,

$$
\sum_{n=1}^{\infty} b_n \sin(n\pi x),
$$

we integrate in a straightforward way to obtain,

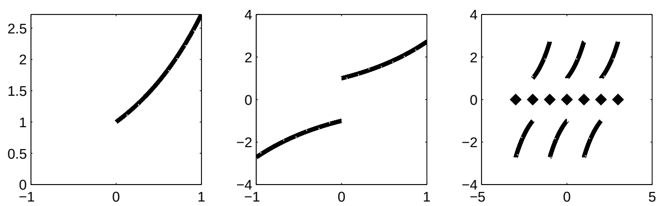
$$
\begin{aligned}
b_n &= \int_{-1}^{1} \tilde{f}_{odd}(x)\sin(n\pi x)\, dx = 2\int_{0}^{1} \tilde{f}_{odd}(x)\sin(n\pi x)\, dx \\
&= 2\int_{0}^{1} 10\sin(n\pi x)\, dx = 20\left[\left(\frac{-1}{n\pi}\right)\cos(n\pi x)\right]_{0}^{1} = 20\left[\left(\frac{-1}{n\pi}\right)\cos(n\pi) + \frac{1}{n\pi}\right] \\
&= \frac{20}{n\pi}\left[1 + (-1)^{n+1}\right].
\end{aligned}
$$

b) Below we plot $f(x) = x$, on the interval $[0, 1]$, (left), the odd extension $\tilde{f}_{odd}(x)$ on the interval $[-1, 1]$ (middle), and the Fourier sine series associated with the periodic extension $\tilde{f}_{odd}(x)$ (right).



Note that since $\tilde{f}_{odd}(x) = x$, the Fourier sine series associated with the periodic extension of this function is the Fourier series derived in question 1 part a). The coefficients were derived in question 2 part b).

c) Below we plot $f(x) = \exp(x)$, on the interval $[0, 1]$, (left), the odd extension $\tilde{f}_{odd}(x)$ on the interval $[-1, 1]$ (middle), and the Fourier sine series associated with the periodic extension $\tilde{f}_{odd}(x)$ (right).



5. The standard Fourier Series is,

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

If $g(x)$ is *even* on $[-L, L]$ we have the property that,

$$\int_{-L}^{0} g(x)\,dx \;=\; \int_{0}^{L} g(x)\,dx.$$

Since $\cos\left(\frac{n\pi x}{L}\right)$ is an even function, the product $g(x)\cos\left(\frac{n\pi x}{L}\right)$ is also even.
Hence, applying the definitions for the Fourier coefficients, we obtain,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} g(x)\,dx = \frac{1}{2L}\int_{-L}^{0} g(x)\,dx + \frac{1}{2L}\int_{0}^{L} g(x)\,dx = \frac{2}{2L}\int_{0}^{L} g(x)\,dx = \frac{1}{L}\int_{0}^{L} g(x)\,dx,$$

$$a_n = \frac{1}{L}\int_{-L}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{L}\int_{-L}^{0} g(x)\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{L}\int_{0}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right) = \frac{2}{L}\int_{0}^{L} g(x)\cos\left(\frac{n\pi x}{L}\right).$$

Now, for any *odd* function $h(x)$ on the interval $[-L, L]$, we have,

$$\int_{-L}^{0} h(x)\,dx \;=\; -\int_{0}^{L} h(x)\,dx.$$

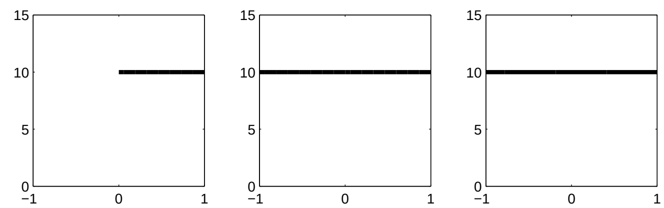
Since the function $\sin\left(\frac{n\pi x}{L}\right)$ is odd, when $g(x)$ is even, the product, $g(x)\sin\left(\frac{n\pi x}{L}\right)$ is also odd. Hence, the remaining Fourier coefficients are,

$$b_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{L}\int_{-L}^{0} g(x)\sin\left(\frac{n\pi x}{L}\right) + \frac{1}{L}\int_{0}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right) = 0.$$

The resulting Fourier series has no 'sine' terms and takes the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

6. a) Below we plot $f(x) = 10$, on the interval $[0, 1]$, (left), the even extension $\tilde{f}_{even}(x)$ on the interval $[-1, 1]$ (middle), and the Fourier cosine series associated with the periodic extension $\tilde{f}_{even}(x)$ (right).

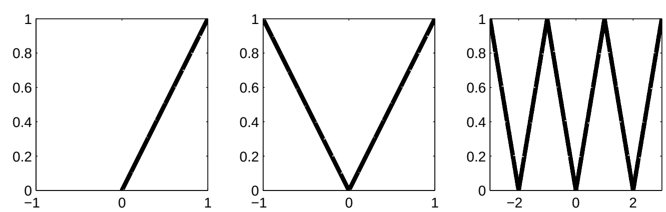


The Fourier cosine series of $\tilde{f}_{even}$ is therefore trivial. We have, $\sum_{n=0}^{\infty} a_n \cos(n\pi x)$ with

$$a_0 = \frac{1}{1}\int_0^1 10\, dx = 10, \quad a_n = \frac{2}{1}\int_0^1 \cos(n\pi x)\, dx = 0 \quad n = 1, 2, \ldots$$

Note: compare the Fourier sine and cosine series associated with the periodic extension of the odd, and even extensions (respectively) of $f(x) = 10$. The answers are very different!

b) Below we plot $f(x) = x$, on the interval $[0, 1]$, (left), the even extension $\tilde{f}_{even}(x)$ to the interval $[-1, 1]$ (middle), and the Fourier cosine series associated with the periodic extension $\tilde{f}_{even}(x)$ (right). To compute the coefficients of the Fourier cosine series,

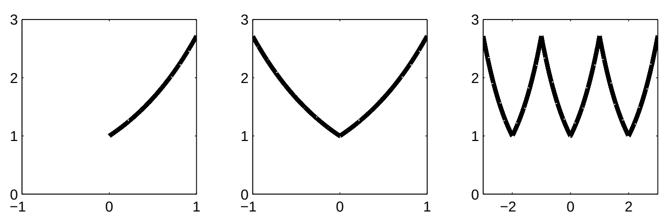


$$\sum_{n=0}^{\infty} a_n \cos(n\pi x),$$

we integrate by parts to obtain, $a_0 = \frac{1}{1}\int_0^1 x\, dx = \frac{1}{2}$ and,

$$\begin{aligned} a_n &= \frac{2}{1}\int_0^1 x\cos(n\pi x)\, dx = 2\left\{\left[\left(\frac{x}{n\pi}\right)\sin(n\pi x)\right]_0^1 - \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\, dx\right\} \\ &= 2\left\{0 - \frac{1}{n\pi}\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^1\right\} = -\frac{2}{n^2\pi^2}\left[1 + (-1)^{n+1}\right], \quad n = 1, 2, \ldots \end{aligned}$$

c) Below we plot $f(x) = \exp(x)$, on the interval $[0, 1]$, (left), the even extension $\tilde{f}_{even}$ to the interval $[-1, 1]$ (middle), and the Fourier cosine series associated with the periodic extension $\tilde{f}_{even}(x)$ (right).



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