1. Note: Sketch the graphs

   a) $f(x) = \ln(x)$ is not pwc on any interval that contains $x = 0$. $f(x)$ is continuous on the open interval $(0, 1)$ but the limit $f(0^+)$ does not exist.

   b) $f(x) = x^{\frac{1}{3}}$ is continuous on the open interval $(0, 1)$ and the limits $f(0^+) = 0$, $f(1^-) = 1$ both exist. Hence $f(x)$ is pwc on $[0, 1]$.

   c) $f(x) = (1 - x)^{-1}$ is continuous on the open interval $(0, 1)$. The limit $f(0^+) = 1$ exists but the limit $f(1^-) =$ does not exist. Hence $f(x)$ is not pwc on $[0, 1]$.

   d) $f(x)$ is continuous on each of the open subintervals $(0, \frac{1}{4})$, $(\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{2}, 1)$. The limits $f(0^+) = 1, f(\frac{1}{4}^-) = 1, f(\frac{1}{4}^+) = 2, f(\frac{1}{2}^-) = 2, f(\frac{1}{2}^+) = 0, f(1^-) = 0$ all exist. Hence $f(x)$ is pwc on $[0, 1]$.

2. To prove that the given functions are *orthogonal* it is necessary to show that

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\, dx = 0, \qquad \text{for } n \neq m.$$

   Using the hint, the integrand simplifies to give

$$\begin{aligned}
& \frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{(n+m)\,\pi x}{L}\right) + \cos\left(\frac{(n-m)\,\pi x}{L}\right)\right) dx \\
= \; & \frac{1}{2}\left[\frac{L}{\pi\,(n+m)}\sin\left(\frac{(n+m)\,\pi x}{L}\right) + \frac{L}{\pi\,(n-m)}\sin\left(\frac{(n-m)\,\pi x}{L}\right)\right]_{-L}^{L} \\
= \; & \frac{1}{2}\left[\frac{L}{\pi\,(n+m)}\sin\left((n+m)\,\pi\right) + \frac{L}{\pi\,(n-m)}\sin\left((n-m)\,\pi\right)\right] \\
- \; & \frac{1}{2}\left[\frac{L}{\pi\,(n+m)}\sin\left(-\,(n+m)\,\pi\right) + \frac{L}{\pi\,(n-m)}\sin\left(-\,(n-m)\,\pi\right)\right] \\
= \; & \frac{L}{\pi\,(n+m)}\sin\left((n+m)\,\pi\right) + \frac{L}{\pi\,(n-m)}\sin\left((n-m)\,\pi\right) = 0,
\end{aligned}$$

   since $\sin((n+m)\,\pi) = 0$ and $\sin((n-m)\,\pi) = 0$ for $n \neq m$ and $n, m = 1, 2, 3 \ldots$

3. The square of the norm of the nth function in the given set is defined via

$$\left\|\cos\left(\frac{n\pi x}{L}\right)\right\|^2 = \int_{-L}^{L}\cos^2\left(\frac{n\pi x}{L}\right)\, dx.$$

   Using identity $\cos^2(A) = (1 + \cos(2A))/2$, the integrand simplifies to

$$\begin{aligned}
\left\|\cos\left(\frac{n\pi x}{L}\right)\right\|^2 &= \frac{1}{2}\int_{-L}^{L} 1 + \cos\left(\frac{2n\pi x}{L}\right)\, dx \\
&= \frac{1}{2}\,[2L] + \frac{1}{2}\left[\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_{-L}^{L} = L + 0 = L.
\end{aligned}$$

   Therefore, $\| \cos\left(\frac{n\pi x}{L}\right) \| = \sqrt{L}$, $n = 1, 2, \ldots$. To normalise the orthogonal functions, we divide each one by its norm, so that the new *orthonormal* set is:

$$\left\{\frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right)\right\}, \; n = 1, 2, \ldots$$

4. Note that the set of functions $\left\{1, \cos\frac{n\pi x}{L}, \sin\frac{m\pi x}{L}\right\}, n, m = 1, 2, \ldots$ is the set of orthogonal functions used in the Fourier Series of a pwc $f(x)$. To show that they do indeed form an orthogonal set it is necessary to prove that:

   (a) $\int_{-L}^{L} 1 \cos\left(\frac{n\pi x}{L}\right) dx = 0, \quad n = 1, 2, \ldots$
   (b) $\int_{-L}^{L} 1 \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad n = 1, 2, \ldots$
   (c) $\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = 0, \quad n \neq m, n, m = 1, 2, \ldots$
   (d) $\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0, \quad n, m = 1, 2, \ldots$
   (e) $\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = 0, \quad n \neq m, n, m = 1, 2, \ldots$

   Notice first that property d) was proved in the lectures (see lecture notes for calculation) and property c) was proved in question 2. Properties a) and b) are trivial since:

$$\int_{-L}^{L} 1 \cos\left(\frac{n\pi x}{L}\right) dx = \frac{L}{n\pi}\left[\sin(n\pi) - \sin(-n\pi)\right] = 0$$
$$\int_{-L}^{L} 1 \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{L}{n\pi}\left[\cos(n\pi) - \cos(-n\pi)\right] = 0$$

   Hence, it remains to prove property e). For $n, m = 1, 2, \ldots$ with $n \neq m$,

$$\begin{aligned}
\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx &= \frac{1}{2}\int_{-L}^{L}\left(\cos\left(\frac{(n+m)\pi x}{L}\right) - \cos\left(\frac{(n-m)\pi x}{L}\right)\right) dx \\
&= \frac{1}{2}\left[\frac{L}{\pi(n+m)}\frac{\sin((n+m)\pi x)}{L}\right]_{-L}^{L} - \frac{1}{2}\left[\frac{L}{\pi(n-m)}\frac{\sin((n-m)\pi x)}{L}\right]_{-L}^{L} \\
&= 0 + 0 = 0.
\end{aligned}$$

5. We show that $f(x) = 1$ and $g(x) = x$ are orthogonal with respect to the weighted inner product by showing that

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx = 0.$$

   You should recognise this integral from 1st year calculus courses. If we make the change of variables $\sin u = x$ so that $\cos u \, du = dx$, then

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin u}{\cos u} \cos u \, du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin u \, du = [-\cos u]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -\cos\frac{\pi}{2} + \cos\frac{\pi}{2} = 0.$$

   Alternatively, it suffices to note that the integrand is an odd function, so the integral over $[-1, 1]$ must be zero.

6. Note: sketching the graphs will help with this question.

   a) In question 1 it was shown that $f(x) = \ln(x)$ is not pwc on $[0, 1]$ since the one-sided limit $f(0^+)$ does not exist. Hence, $f(x)$ is not pws on $[0, 1]$.

   b) In question 1 we established that $f(x) = x^{\frac{1}{3}}$ is pwc on $[0, 1]$. It remains to check whether $f'(x)$ is also pwc. We have $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. Notice that the limit $f'(1^-) = \frac{1}{3}$ exists but the other limit $f'(0^+)$ does not exist. Hence $f(x)$ is not pws on $[0, 1]$.

   c) $f(x) = \frac{1}{x}$ is continuous on the open interval $(0, 1)$ but since the limit $f(0^+)$ does not exist, $f(x)$ is not pwc on $[0, 1]$. Hence it is not pws on $[0, 1]$.

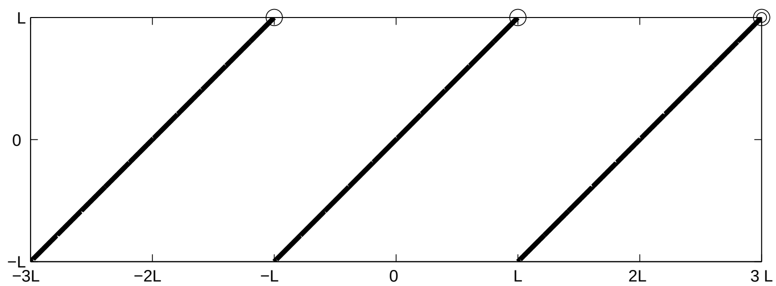
d) In question 1 we established that $f(x)$ is pwc on $[0,1]$. It remains to check the first derivatives. We have,

$$f^{'}(x) = \begin{cases} 0 & x < \frac{1}{4} \\ 0 & \frac{1}{4} < x < \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases} .$$

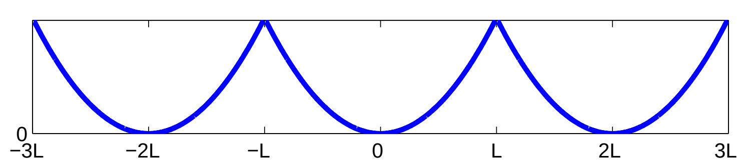
All the one-sided limits $f^{'}(0^{+}), f^{'}(\frac{1}{4}^{-}), f^{'}(\frac{1}{4}^{+}), f^{'}(\frac{1}{2}^{-}), f^{'}(\frac{1}{2}^{+}), f^{'}(1^{-})$ exist and are equal to zero. $f^{'}(x)$ is pwc on $[0,1]$. hence $f(x)$ is pws on $[0,1]$.

7. **Note:** When drawing periodic extensions starting with $[-L, L]$ in the case that $f(L) \neq f(-L)$, we are faced with the decision whether to keep the left value $\tilde{f}(-L+2Ln) = f(-L)$ or the right value $\tilde{f}(L+2Ln) = f(L)$, for all integer n. In the lecture notes we have used the first option, but the second is equally valid as long as the resulting periodic extension is consistent with this choice.

a) Consider $f(x) = x$, $x \in [-L, L]$. The periodic extension $\tilde{f}(x)$ on the interval $[-3L, 3L]$ is:

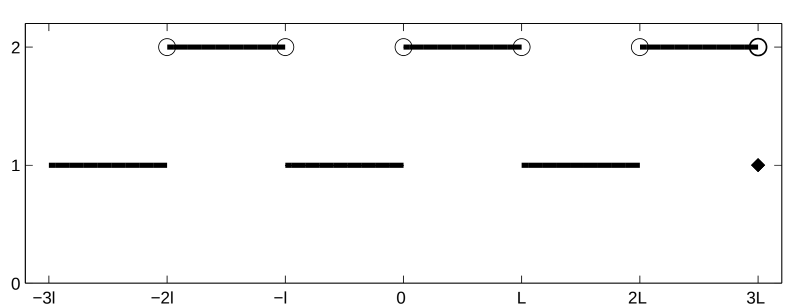


b) Consider $f(x) = x^2$, $x \in [-L, L]$. The periodic extension $\tilde{f}(x)$ on the interval $[-3L, 3L]$ is:



c) Consider $f(x) = \begin{cases} 1 & -L \leq x \leq 0 \\ 2 & 0 < x \leq L \end{cases}$

The periodic extension $\tilde{f}(x)$ on the interval $[-3L, 3L]$ is:



Note that in the above graphs, an open circle '$\circ$' means that the function is discontinuous at that value of x. $f(x)$ takes the value indicated by a closed circle.