1. Recall, from the lecture notes, that we can find a unit normal vector to a surface expressed as $z = f(x, y)$ via
$$\hat{\mathbf{n}} = \frac{-\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}.$$
Note that the question just asks for *a* unit vector to the given surface and does not specify a direction.

   a) With $z = f(x, y) = 2 - x - y$, we have $\frac{\partial f}{\partial x} = -1$, $\frac{\partial f}{\partial y} = -1$, and so:
$$\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

   b) With $z = f(x, y) = \left(1 - x^2\right)^{\frac{1}{2}}$ we have $\frac{\partial f}{\partial x} = -\frac{x}{z}$, $\frac{\partial f}{\partial y} = 0$, and so:
$$\hat{\mathbf{n}} = \frac{\frac{x}{z}\mathbf{i} - 0\mathbf{j} + \mathbf{k}}{\sqrt{1 + \frac{x^2}{z^2}}} = \frac{x\mathbf{i} + z\mathbf{k}}{\sqrt{z^2 + x^2}} = x\mathbf{i} + z\mathbf{k}$$
   since $x^2 + z^2 = 1$.

   c) With $z = f(x, y) = \left(1 - x^2 - y^2\right)^{\frac{1}{2}}$ we have $\frac{\partial f}{\partial x} = -\frac{x}{z}$, $\frac{\partial f}{\partial y} = -\frac{y}{z}$, and so:
$$\hat{\mathbf{n}} = \frac{\frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2 + x^2 + y^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

2. Note that this question is really the same as 1 c). The formula for the surface of the unit sphere is $z^2 + y^2 + x^2 = 1$. For points on the surface of the upper hemisphere we have $z = +\sqrt{1 - x^2 - y^2}$, and the lower hemisphere is given by $z = -\sqrt{1 - x^2 - y^2}$. Hence a unit normal vector to the upper hemisphere is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ [as calculated in 1 c)] and it should be obvious that this vector is pointing in the outward direction from this surface. (Draw as sketch to help you visualise this.) To obtain a unit normal vector to the lower hemisphere, repeat the above calculation with $z = -\sqrt{1 - x^2 - y^2}$, to obtain $\frac{\partial f}{\partial x} = -\frac{x}{z}$, $\frac{\partial f}{\partial y} = -\frac{y}{z}$, and so, again:
$$\hat{\mathbf{n}} = \frac{\frac{x}{z}\mathbf{i} + \frac{y}{z}\mathbf{j} + \mathbf{k}}{\sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{z^2 + x^2 + y^2}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$
Once more it should be obvious that this vector points outward from the surface of the hemisphere.
Putting the two halves together, we have shown that the outward unit normal vector to the unit sphere is the 3-dimensional radial vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

3. In each case we have $\int\int_S G(x, y, z)\, dS = \int\int_D G(x, y, f(x, y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy$, where D is the projection of the surface S onto the x-y plane.

   a) With S given by the equation $z = 1 - x - y = f(x, y)$ we have:
$$\int\int_S G(x, y, z)\, dS = \int\int_D (1 - x - y)\sqrt{1 + (-1)^2 + (-1)^2}\, dxdy = \sqrt{3}\int\int_D 1 - x - y\, dxdy.$$
   The intersection of S with the x-y plane in the first octant is the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$. Hence the integral becomes:
$$\sqrt{3}\int_{x=0}^{1}\int_{y=0}^{1-x}(1 - x - y)\, dydx = \sqrt{3}\int_{x=0}^{1}\left(\frac{x^2}{2} - x + \frac{1}{2}\right) dx = \frac{\sqrt{3}}{6}.$$

b) With S given by the equation $z = x^2 + y^2 = f(x, y)$ we have:

$$\begin{aligned}
\int\int_S G(x,y,z)\,dS &= \int\int_D \frac{1}{1+4\left(x^2+y^2\right)}\sqrt{1+(2x)^2+(2y)^2}\,dxdy \\
&= \int\int_D \frac{1}{1+4\left(x^2+y^2\right)}\sqrt{1+4x^2+4y^2}\,dxdy \\
&= \int\int_D \left(1+4\left(x^2+y^2\right)\right)^{-\frac{1}{2}}\,dxdy.
\end{aligned}$$

Now, the projection of S onto the x-y plane is the circle $x^2 + y^2 = 1$. Posing the integral in x and y coordinates is messy. However, it can be simplified if we use polar coordinates. **Recall that when we change variables in the integral, the element of area also changes:** $dxdy = r\,drd\theta$. Hence:

$$\begin{aligned}
\int\int_S G(x,y,z)\,dS &= \int_{\theta=0}^{2\pi}\int_0^1 \left(1+4r^2\right)^{-\frac{1}{2}}\,r\,drd\theta \\
&= \int_{\theta=0}^{2\pi}\left[\frac{1}{4}\left(1+4r^2\right)^{\frac{1}{2}}\right]_{r=0}^{1}\,d\theta \\
&= \int_{\theta=0}^{2\pi}\frac{1}{4}\left(\sqrt{5}-1\right)d\theta = \left[\frac{\theta}{4}\left(\sqrt{5}-1\right)\right]_{\theta=0}^{2\pi} = \frac{\pi}{2}\left(\sqrt{5}-1\right).
\end{aligned}$$

4. To evaluate $\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS$, we first find $\mathbf{n}$ as in question 1, then obtain $\mathbf{F}\cdot\mathbf{n}$ as a function of x and y, and then proceed as in question 3.

a) Rearranging the expression for the surface gives $z = 1 - \frac{x}{2} - \frac{y}{2} = f(x, y)$. Using the formula in question 1 to obtain $\mathbf{n}$ gives:

$$\mathbf{n} = \frac{1}{\sqrt{6}}\left(\mathbf{i}+\mathbf{j}+2\mathbf{k}\right).$$

Hence with $\mathbf{F} = x\mathbf{i} - z\mathbf{k}$, we obtain $\mathbf{F}\cdot\mathbf{n} = \frac{1}{\sqrt{6}}(x-2z) = \frac{1}{\sqrt{6}}(2x+y-2)$ and

$$\sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{2}.$$

Hence, the surface integral then becomes:

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS = \int\int_D \frac{1}{\sqrt{6}}(2x+y-2)\frac{\sqrt{6}}{2}\,dxdy$$

where D is the projection of the surface S onto the x-y plane. In this case, this is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 2$. Hence we have:

$$\begin{aligned}
\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS &= \frac{1}{2}\int_{x=0}^{2}\int_{y=0}^{2-x} 2x+y-2\,dydx \\
&= \frac{1}{2}\int_{x=0}^{2}\left[2xy-\frac{y^2}{2}-2y\right]_0^{2-x}dx = \frac{1}{2}\int_{x=0}^{2}\left(4x-\frac{3x^2}{2}-2\right)dx = 0.
\end{aligned}$$

b) Rearranging the expression for the surface gives $z = \pm\sqrt{a^2 - x^2 - y^2} = f(x, y)$. The positive root gives the formula for the surface of the upper hemisphere of the sphere (of radius a) and the negative root gives the formula for the surface of the upper hemisphere of the sphere. Using the formula in question 1 (and the arguments in question 2) to obtain $\mathbf{n}$ gives:

$$\mathbf{n} = \frac{\frac{x}{z}\mathbf{i}+\frac{y}{z}\mathbf{j}+\mathbf{k}}{\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{\sqrt{z^2+x^2+y^2}} = \frac{1}{a}\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right)$$

Hence with $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, we obtain $\mathbf{F} \cdot \mathbf{n} = \frac{1}{a}\left(x^2 + y^2 + z^2\right) = \frac{a^2}{a} = a$ The surface integral then becomes:

$$\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS \quad = \quad a \int\int_S 1\, dS$$

which is a multiplied by the surface area of the sphere. The formula for the surface area of a sphere of radius a is $4\pi a^2$. Hence the result is $4\pi a^3$.

5. The Divergence Theorem states that for the given vector field $\mathbf{F}$, a three-dimensional volume V and the corresponding surface S of that volume,

$$\int\int\int_V \nabla \cdot \mathbf{F}\, dV \; = \; \int\int_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where $\mathbf{n}$ is the unit outward normal vector to the surface in question.

a) For $\mathbf{F} = (y - x)\mathbf{i} + (y - z)\mathbf{j} + (x - y)\mathbf{k}$, and $V = [0, 1] \times [0, 1] \times [0, 1]$, we have:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = -1 + 1 + 0 = 0.$$

Hence the left-hand side of the equation in the Divergence Theorem is trivial:

$$\int\int\int_V \nabla \cdot \mathbf{F}\, dV = \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} 0 \; dV \; = \; 0.$$

To evaluate the right-hand side, it is easiest to think of the surface of the cube as being composed of 6 faces (two in the x-y plane, two in the y-z plane and two in the x-z plane).
Let the first face be the face that coincides with the plane $z = 1$. Then $\mathbf{n}_1 = \mathbf{k}$ and $\mathbf{F} \cdot \mathbf{n}_1 = (x - y)$. Choose the second face to be the face that coincides with the plane $z = 0$. Then $\mathbf{n}_2 = -\mathbf{k}$ and $\mathbf{F} \cdot \mathbf{n}_2 = (y - x)$. Choose the third face to be the face that coincides with the plane $x = 1$. Then $\mathbf{n}_3 = \mathbf{i}$ and $\mathbf{F} \cdot \mathbf{n}_3 = (y - x)$. Choose the fourth face to be the face that coincides with the plane $x = 0$. Then $\mathbf{n}_4 = -\mathbf{i}$ and $\mathbf{F} \cdot \mathbf{n}_4 = (x - y)$. Choose the fifth face to be the face that coincides with the plane $y = 0$. Then $\mathbf{n}_5 = -\mathbf{j}$ and $\mathbf{F} \cdot \mathbf{n}_5 = (z - y)$. Choose the sixth face to be the face that coincides with the plane $y = 1$. Then $\mathbf{n}_6 = \mathbf{j}$ and $\mathbf{F} \cdot \mathbf{n}_6 = (y - z)$. Hence the surface integral becomes:

$$\begin{aligned}
\int\int_S \mathbf{F} \cdot \mathbf{n}\, dS \;&=\; \int\int_{face1} (x - y)\; dS + \int\int_{face2} (y - x)\; dS + \int\int_{face3} (y - x)\; dS \\
&+\; \int\int_{face4} (x - y)\; dS + \int\int_{face5} (z - y)\; dS + \int\int_{face6} (y - z)\; dS \\
&=\; \int_{x=0}^{1}\int_{y=0}^{1} (x - y)\; dydx + \int_{x=0}^{1}\int_{y=0}^{1} (y - x)\; dydx + \int_{y=0}^{1}\int_{z=0}^{1} (y - x)\; dzdy \\
&+\; \int_{y=0}^{1}\int_{z=0}^{1} (x - y)\; dzdy + \int_{x=0}^{1}\int_{z=0}^{1} (z - y)\; dzdx + \int_{x=0}^{1}\int_{z=0}^{1} (y - z)\; dzdx \\
&=\; 0 + 0 - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 0.
\end{aligned}$$

Hence the theorem is satisfied.

b) For $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and V equal to the sphere of radius a (centred at the origin) we have:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = 1 + 1 + 1 = 3.$$

Hence the left-hand side of the equation in the Divergence Theorem is trivial:

$$\int\int\int_V \nabla\cdot\mathbf{F}\,dV = 3\int\int\int_V 1\,dV$$

which is three times the volume of the sphere. The integral can be calculated efficiently using spherical coordinates as in question 15, exercise sheet 9 and is the well-known formula: $\frac{4\pi a^3}{3}$. Hence

$$\int\int\int_V \nabla\cdot\mathbf{F}\,dV = 4\pi a^3.$$

Since S is the surface of the sphere, we known from question 4 b) that:

$$\mathbf{n} = \frac{1}{a}\left(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\right)$$

and $\mathbf{F}\cdot\mathbf{n} = \frac{1}{a}\left(x^2+y^2+z^2\right) = \frac{a^2}{a} = a$ The surface integral on the right-hand side of the Divergence Theorem then becomes:

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,dS \quad = \quad a\int\int_S 1\,dS$$

which is a multiplied by the surface area of the sphere. The formula for the surface area of a sphere of radius a is $4\pi a^2$. Hence the result is $4\pi a^3$. This agrees with the value of the volume integral. Hence the theorem is verified.

6. Stokes' Theorem states

$$\oint_C \mathbf{F}\cdot\mathbf{t}\,ds = \int\int_S \left(\nabla\times\mathbf{F}\right)\cdot\mathbf{n}\,dS$$

a) In this example, C is the perimeter of the given triangle (which is a portion of the plane $x+y+z=1$). Let C_1 be the portion of C joining the vertex $(0,0,1)$ to $(1,0,0)$. Let C_2 be the portion of C joining the vertex $(1,0,0)$ to $(0,1,0)$.Let C_3 be the portion of C joining the vertex $(0,1,0)$ to $(0,0,1)$.

First, we have:

$$\mathbf{F}\cdot\mathbf{t} = \left(y+y^2\right)\mathbf{k}\cdot\left(\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}\right) = \left(y+y^2\right)\frac{dz}{ds},$$

and so

$$\oint_C \mathbf{F}\cdot\mathbf{t}\,ds = \oint_C \left(y+y^2\right)dz = \int_{C_1}\left(y+y^2\right)dz + \int_{C_2}\left(y+y^2\right)dz + \int_{C_3}\left(y+y^2\right)dz.$$

We consider C as being composed of the three sides of the triangle separately and compute the line integral along each one by converting the variables to the arc length parameter,s.

Consider first the directed line C_1 joining the vertices $(0,0,1)$ and $(1,0,0)$. The arc length parameter s is zero at the first vertex and equal to $\sqrt{2}$ (the length of the line) at the second vertex. Along this line, $y=0$, $x=\frac{s}{\sqrt{2}}$ and $z=1-\frac{s}{\sqrt{2}}$. Hence,

$$\int_{C_1}\left(y+y^2\right)dz = \int_{C_1}\left(y+y^2\right)\frac{dz}{ds}\,ds = \int_{s=0}^{\sqrt{2}}\left(y(s)+y(s)^2\right)\frac{dz}{ds}\,ds = \int_{s=0}^{\sqrt{2}} 0\left(-\frac{1}{\sqrt{2}}\right)ds = 0.$$

Now, consider the directed line C_2 joining the vertex $(1,0,0)$ to $(0,1,0)$. The arc length parameter s is zero at the first vertex and equal to $\sqrt{2}$ (the length of the line) at the second vertex. Along this line, $z=0$, $y=\frac{s}{\sqrt{2}}$ and $x=1-\frac{s}{\sqrt{2}}$. Hence,

$$\int_{C_2} \left(y + y^2\right) dz = \int_{s=0}^{\sqrt{2}} \left(y(s) + y(s)^2\right) \frac{dz}{ds}\, ds = \int_{s=0}^{\sqrt{2}} \left(y(s) + y(s)^2\right)(0)\, ds = 0.$$

Finally, consider the directed line C_3 from the vertex $(0,1,0)$ to $(0,0,1)$. The arc length parameter s is zero at the first vertex and equal to $\sqrt{2}$ (the length of the line) at the second vertex. Along this line, $x = 0$, $z = \frac{s}{\sqrt{2}}$ and $y = 1 - \frac{s}{\sqrt{2}}$. Hence,

$$\int_{C_3} \left(y + y^2\right) dz = \int_{s=0}^{\sqrt{2}} \left(y(s) + y(s)^2\right) \frac{dz}{ds}\, ds = \int_{s=0}^{\sqrt{2}} \left(1 - \frac{s}{\sqrt{2}}\right) + \left(1 - \frac{s}{\sqrt{2}}\right)^2 \left(\frac{1}{\sqrt{2}}\right) ds$$

$$= \frac{1}{\sqrt{2}} \int_{s=0}^{\sqrt{2}} 2 - \frac{3s}{\sqrt{2}} + \frac{s^2}{2}\, ds = \frac{1}{\sqrt{2}} \left[2s - \frac{3s^2}{2\sqrt{2}} + \frac{s^3}{6}\right]_{s=0}^{\sqrt{2}} = \frac{5}{6}.$$

For the second integral, we first compute the curl of $\mathbf{F}$ via:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & y + y^2 \end{vmatrix} = (1 + 2y)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + [0 - (-0)]\,\mathbf{k} = (1 + 2y)\,\mathbf{i}.$$

The outer unit normal to the surface S (which can be written as $z = 1 - x - y$) is found using the standard formula:

$$\mathbf{n} = \frac{-(-1)\mathbf{i} - (-1)\mathbf{j} + \mathbf{k}}{\sqrt{1 + 1 + 1}} = \frac{1}{\sqrt{3}} (\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Hence,

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \frac{1}{\sqrt{3}} \int\int_S 1 + 2y\, dS.$$

The surface integral may be reduced to a double integral over the projection D of the triangle onto the $x - y$ plane (in this case, the triangle bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$. (Sketch this to see it).

$$\frac{1}{\sqrt{3}} \int\int_S 1 + 2y\, dS = \frac{1}{\sqrt{3}} \int\int_D (1 + 2y) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

$$= \frac{1}{\sqrt{3}} \int\int_D (1 + 2y) \sqrt{1 + 1^2 + 1^2}\, dxdy = \int\int_D 1 + 2y dxdy$$

$$= \int_{y=0}^{1} \int_{x=0}^{1-y} 1 + 2y\, dxdy = \frac{5}{6}.$$

b) In this example, C is the circle $x^2 + y^2 = 4$, $z = 0$, and S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

First, we have:

$$\mathbf{F} \cdot \mathbf{t}\, ds = (-y\mathbf{i} + x\mathbf{j}) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) = -y\frac{dx}{ds} + x\frac{dy}{ds}, \quad \Rightarrow \oint_C \mathbf{F} \cdot \mathbf{t}\, ds = \oint_C -ydx + xdy.$$

Although it is standard to parameterise line integrals using the arc-length parameter, s, the curve C in this question is a circle and may be represented parametrically by $x = 2\cos\theta$, $y = 2\sin\theta$, $z = 0$ (where $0 \leq \theta \leq 2\pi$). Then

$$\frac{dx}{d\theta} = -2\sin\theta, \qquad \frac{dy}{d\theta} = 2\cos\theta, \qquad \frac{dz}{d\theta} = 0$$

so

$$\oint_C \mathbf{F}\cdot\mathbf{t}\,ds = \int_0^{2\pi} (-2\sin\theta)\,(-2\sin\theta d\theta) + (2\cos\theta)\,(2\cos\theta d\theta) = 4\int_0^{2\pi} d\theta = 8\pi.$$

For the second integral, we first compute the curl of $\mathbf{F}$ via:

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + [1-(-1)]\,\mathbf{k} = 2\mathbf{k}.$$

The outer unit normal to the surface S (which can be written as $z = +\sqrt{4-x^2-y^2}$) is found as in question 4b) with $a = 2$):

$$\mathbf{n} = \frac{1}{2}\left(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}\right).$$

Hence,

$$\int\int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \int\int_S z\,dS.$$

The surface integral may be reduced to a double integral over a plane area (the disc $x^2+y^2 \leq 4$) by using the standard formula:

$$\int\int_S z\,dS = \int\int_{x^2+y^2\leq 4} z(x,y)\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$$

$$= \int\int_{x^2+y^2\leq 4} z(x,y)\sqrt{1+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\,dxdy = \int\int_{x^2+y^2\leq 4} z\sqrt{\frac{z^2}{z^2}+\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2}\,dxdy$$

$$= \int\int_{x^2+y^2\leq 4} z\,\frac{1}{z}\,2\,dxdy = \int\int_{x^2+y^2\leq 4} 2\,dxdy = 2\pi 2^2 = 8\pi.$$

7. (Sketch omitted here). Let S_1, S_2, S_3, S_4 and S_5 be the surfaces associated with the planes $y = 0$, $y = 2$, $x = 1$, $x = 0$ and $z = 1$, respectively. Then

$$\hat{\mathbf{n}}_1 = -\mathbf{j}, \quad \hat{\mathbf{n}}_2 = \mathbf{j}, \quad \hat{\mathbf{n}}_3 = \mathbf{i}, \quad \hat{\mathbf{n}}_4 = -\mathbf{i}, \quad \hat{\mathbf{n}}_5 = \mathbf{k}.$$

Let C_1, C_2, C_3 and C_4 denote the line segements associated with $y = 0$, $x = 1$, $y = 2$ and $x = 0$ respectively. Then

$$\hat{\mathbf{t}}_1 = \mathbf{i}, \quad \hat{\mathbf{t}}_2 = \mathbf{j}, \quad \hat{\mathbf{t}}_3 = -\mathbf{i}, \quad \hat{\mathbf{t}}_4 = -\mathbf{j}.$$

We compute the surface integral first. For the given $\mathbf{F}$,

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x^2 z & xy \end{vmatrix} = \left(x+x^2\right)\mathbf{i} - y\mathbf{j} - (2xz+1)\mathbf{k}.$$

On S_1, $(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_1 = y = 0$. Hence, $\int\int_{S_1}(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_1 dS_1 = 0$.

On S_2, $(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_2 = -y = -2$. Hence, $\int\int_{S_2}(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_2 dS_2 = -2|S_2| = -2$.

On S_3, $(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_3 = x + x^2 = 1 + 1 = 2$. Hence, $\int\int_{S_3}(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_3 dS_3 = 2|S_3| = 4$.

On S_4, $(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_4 = -x^2 - x = 0 - 0 = 0$. Hence, $\int\int_{S_4}(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_4 dS_4 = 0$.

On S_5, $(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_5 = -2xz - 1 = -2x - 1$. Hence,

$$\int\int_{S_5}(\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}_5 dS_5 = \int_{y=0}^{2}\int_{x=0}^{1} -2x - 1 dxdy = -4.$$

Adding these tells us that the surface integral in Stokes Theorem is:

$$\int\int_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}} dS = -2.$$

We can break the line integral up into four parts:

$$\oint_C \mathbf{F} \cdot \mathbf{t}\, ds = \sum_{i=1}^{4} \int_{C_i} \mathbf{F} \cdot \hat{\mathbf{t}}_i\, ds.$$

On C_1, $\mathbf{F} \cdot \hat{\mathbf{t}}_1 = y = 0$ so $\int_{C_1} \mathbf{F} \cdot \hat{\mathbf{t}}_1\, ds = \int_{s=0}^{1} 0\, ds = 0$.
On C_2, $\mathbf{F} \cdot \hat{\mathbf{t}}_2 = -x^2 z = 0$ so $\int_{C_2} \mathbf{F} \cdot \hat{\mathbf{t}}_2\, ds = \int_{s=0}^{1} 0\, ds = 0$.
On C_3, $\mathbf{F} \cdot \hat{\mathbf{t}}_3 = -y = -2$ so $\int_{C_3} \mathbf{F} \cdot \hat{\mathbf{t}}_3\, ds = \int_{s=0}^{1} -2\, ds = -2$.
On C_4, $\mathbf{F} \cdot \hat{\mathbf{t}}_4 = x^2 z = 0$ so $\int_{C_4} \mathbf{F} \cdot \hat{\mathbf{t}}_4\, ds = \int_{s=0}^{1} 0\, ds = 0$.

Hence,

$$\oint_C \mathbf{F} \cdot \mathbf{t}\, ds = -2.$$

This matches the surface integral above and so the theorem is satisfied.