Cartesian, Cylindrical and Spherical Co-ordinates

1. Using the relations $x = r\cos\theta$, $y = r\sin\theta$ and $z = z$, we obtain the following sets of Cartesian coordinates:
   a) $(r,\theta,z) = \left(2, \frac{2\pi}{3}, 2\right) \to (x,y,z) = \left(-1, \sqrt{3}, 2\right)$
   b) $(r,\theta,z) = \left(\pi, \frac{\pi}{2}, 1\right) \to (x,y,z) = (0,\pi,1)$
   c) $(r,\theta,z) = (3,0,-6) \to (x,y,z) = (3,0,-6)$

2. Using the relations $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$, we obtain the following sets of Cartesian coordinates:
   a) $(\rho,\theta,\phi) = (3,0,\pi) \to (x,y,z) = (0,0,-3)$
   b) $(\rho,\theta,\phi) = \left(2, \frac{\pi}{4}, \frac{\pi}{3}\right) \to (x,y,z) = \left(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\right)$
   c) $(\rho,\theta,\phi) = \left(3, 0, \frac{2\pi}{3}\right) \to (x,y,z) = \left(\frac{3\sqrt{3}}{2}, 0, -\frac{3}{2}\right)$

3. Using the relations: $r = \sqrt{x^2+y^2}, z = z$ and

$$\theta = \begin{cases} \tan^{-1}\frac{y}{x} & \text{if } x>0, y\geq 0 \\ \tan^{-1}\frac{y}{x} + 2\pi & \text{if } x>0, y<0 \\ \tan^{-1}\frac{y}{x} + \pi & \text{if } x<0 \\ \frac{\pi}{2} & \text{if } x=0, y>0 \\ \frac{3\pi}{2} & \text{if } x=0, y<0 \\ \text{undefined} & \text{if } x=0, y=0 \end{cases}$$

   we obtain the following sets of cylindrical coordinates:
   i) a) $(x,y,z) = (-1,0,2) \to (r,\theta,z) = (1,\pi,2)$
   b) $(x,y,z) = (-1,\sqrt{3},13) \to (r,\theta,z) = (2,\frac{2\pi}{3},13)$
   c) $(x,y,z) = (0,2,-1) \to (r,\theta,z) = (2,\frac{\pi}{2},-1)$

   and using the relations

$$\rho = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{y}{x}$$

   (following the same rules as in part i) and

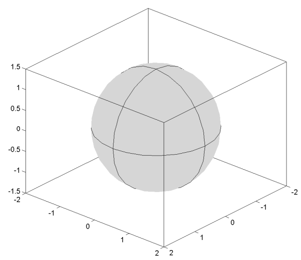
$$\phi = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad 0 \leq \phi \leq \pi,$$

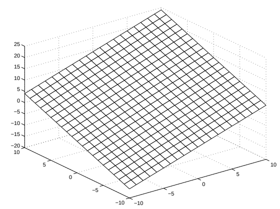
   we obtain the following sets of spherical coordinates:
   ii) a) $(x,y,z) = (-1,0,2) \to (\rho,\theta,\phi) = (\sqrt{5}, 180^o, 26.57^o)$
   b) $(x,y,z) = (-1,\sqrt{3},13) \to (\rho,\theta,\phi) = (\sqrt{173}, 120^o, 8.75^o)$
   c) $(x,y,z) = (0,2,-1) \to (\rho,\theta,\phi) = (\sqrt{5}, 90^o, 116.57^o)$

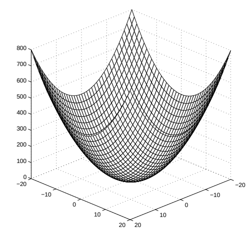
4. a) $x^2 + y^2 + z^2 = 2$ represents a sphere of radius $\sqrt{2}$, centered at the origin $(0, 0, 0)$.

b) $z = 4 + x + y$ is a plane with intercepts : $(0, 0, 4)$, $(0, -4, 0)$ and $(-4, 0, 0)$ and normal vector $-i - j + k$ (since $-x - y + z = 4$, the plane). It is plotted below for the range of values $(-10 \leq x \leq 10, -10 \leq y \leq 10)$ but extends to infinity.

c) $z = x^2 + y^2$ is shown in the right-hand plot below. It has the exotic name of "elliptic paraboloid." Note that the $z \geq 0$.



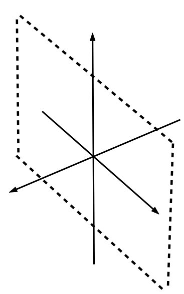


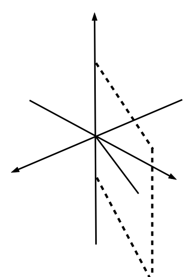


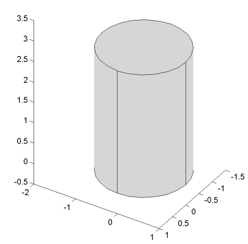
d) $r \cos\theta = 3$. Converting to Cartesian co-ordinates we obtain the plane $x = 3$.

e) $\theta = \frac{\pi}{3}$ is a half plane that intersects the xy plane.

f) $r^2 = 4 \Rightarrow r = \pm 2$. However, r is by definition positive so the result is a cylinder of radius 2, having the z axis as the axis of symmetry and infinite length.



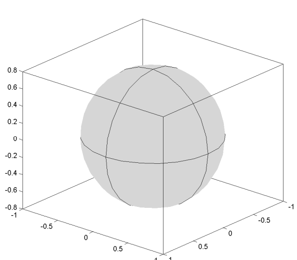


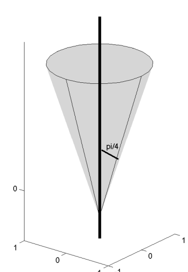


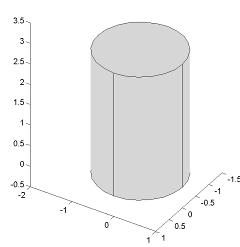
g) $\rho = \frac{\pi}{4}$ is a sphere of radius $\frac{\pi}{4}$ centered at the origin.

h) $\phi = \frac{\pi}{4}$ is a half-cone of arbitrary height. The z-axis is the axis of symmetry and the angle of inclination of the outer walls to the z-axis is $\frac{\pi}{4}$ with tip at the origin

i) Converting to Cartesian coordinates, with $\rho \sin\phi = 2$, we obtain $x = 2\cos\theta$, $y = 2\sin\theta$, $z = \frac{2}{\tan\phi}$. Hence, $r = \sqrt{(x^2 + y^2) = 4}$ and so we recover the equation for a cylinder of radius $r = 2$.







5. We require $x = r\cos\theta = r$ and $y = r\sin\theta = \theta$ (Note that $z = z$ when converting between Cartesian and cylindrical coordinates). To obtain $x = r$, θ must satisfy $\cos\theta = 1$ ie $\theta = 0 \pm 2n\pi$, $n = 1, 2, \ldots$ Now if $\theta = \pm 2n\pi$ we have $y = r\sin(\pm 2n\pi) = 0 \neq \theta$. Hence only the choice $\theta = 0$ is correct giving $y = r\sin(0) = 0 = \theta$. Hence the desired set of points have co-ordinates of the form $(x, 0, z) = (r, 0, z)$. Since $r \geq 0$ and $x = r$, we must also have $x \geq 0$. In Cartesian notation, the set of points $\{(x, 0, z) \mid x \geq 0, -\infty < z < \infty\}$ has the same co-ordinates when written in cylindrical notation. Geometrically, this is half of the xz-plane with $x \geq 0$.

Partial differentiation

6. a) $z_x = \frac{y}{x}$ and $z_y = \ln x$ b) $z_x = 2xe^{x^2-y^2}$ and $z_y = -2ye^{x^2-y^2}$
c) $z_x = -2\sin(x+y)\sin(2x-y) + \cos(x+y)\cos(2x-y)$, and
$z_y = \sin(x+y)\sin(2x-y) + \cos(x+y)\cos(2x-y)$

7. a) Setting $u(x,y) = x^2 + y^2$ and differentiating gives $u_x = 2x$, $u_{xx} = 2$, $u_y = 2y$, $u_{yy} = 2$. Therefore, $u_{xx} + u_{yy} = 2 + 2 = 4$ and so $u(x,y)$ does not satisfy Laplace's equation.

b) Setting $u(x,y) = \ln\sqrt{x^2+y^2}$ and differentiating gives:

$$u_x = \frac{x}{x^2+y^2}, \quad u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}, \quad u_y = \frac{y}{x^2+y^2}, \quad u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2},$$

Adding the second derivatives gives $u_{xx} + u_{yy} = 0$. Hence, this $u(x,y)$ is also a solution to Laplace's equation.

c) Setting $u(x,y) = e^{-x}\cos y - e^{-y}\sin x$ and differentiating gives

$$u_x = -e^{-x}\cos y - e^{-y}\cos x, \quad u_{xx} = e^{-x}\cos y + e^{-x}\sin x,$$

$$u_y = -e^{-x}\sin y + e^{-y}\sin x, \quad u_{yy} = -e^{-x}\cos y - e^{-y}\sin x,$$

Adding the second derivatives gives $u_{xx} + u_{yy} = 0$. Hence, this $u(x,y)$ is a solution to the Laplace's equation.

8. a) Setting $u(x,t) = \sin(x-t) + \ln(x+t)$, and differentiating gives,

$$u_x = \cos(x-t) + \frac{1}{(x+t)}, \quad u_{xx} = -\sin(x-t) - (x+t)^{-2}$$

$$u_t = -\cos(x-t) + \frac{1}{(x+t)}, \quad u_{tt} = -\sin(x-t) - (x+t)^{-2}$$

Hence, $u_{xx} = u_{tt}$ as required. $u(x,t)$ is a solution to the 1D wave equation.

b) Setting $u(x,t) = (x-t)^{-2} + 2\cos(x+t)$, and differentiating gives,

$$u_x = -2(x-t)^{-3} - 2\sin(x+t), \quad u_{xx} = 6(x-t)^{-4} - 2\cos(x+t)$$

$$u_t = +2(x-t)^{-3} - 2\sin(x+t), \quad u_{tt} = +6(x-t)^{-4} - 2\cos(x+t).$$

Hence, $u_{xx} = u_{tt}$ as required. $u(x,t)$ is a solution to the 1D wave equation.

c) Setting $u(x,t) = \exp(x+t) + \exp(x-t)$ and differentiating gives,

$$u_x = \exp(x+t) + \exp(x-t), \quad u_{xx} = \exp(x+t) + \exp(x-t)$$

$$u_t = \exp(x+t) - \exp(x-t), \quad u_{tt} = \exp(x+t) + \exp(x-t).$$

Hence, $u_{xx} = u_{tt}$ as required. $u(x,t)$ is a solution to the 1D wave equation.

Note that each of these solutions to the one-dimensional wave equation has the generic form $f(x+t) + g(x-t)$, where $f(\cdot)$ and $g(\cdot)$ are arbitrary functions.

9. Applying the chain rule, we have:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} &&= \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} &&= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} \\
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial z} &&= \frac{\partial u}{\partial z}
\end{aligned}$$

To obtain the first partial derivatives $\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial y}$, we use the expressions:

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

to obtain:

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} 2x = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta \\
\frac{\partial r}{\partial y} &= \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}} 2y = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta \\
\frac{\partial \theta}{\partial x} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \\
\frac{\partial \theta}{\partial y} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}.
\end{aligned}$$

Note that r and θ need to be written as functions of x and y to perform the differentiation. It is not ok to write x and y as functions of r and θ and find $\frac{\partial x}{\partial r}$ etc. In general, $\frac{\partial r}{\partial x} \neq \left(\frac{\partial x}{\partial r}\right)^{-1}$. Substituting in the above derivatives in the chain rule, the first derivatives are:

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r} \\
\frac{\partial u}{\partial y} &= \frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r} \\
\frac{\partial u}{\partial z} &= \frac{\partial u}{\partial z}
\end{aligned}$$

To obtain the second derivatives, we differentiate again (messy) to obtain:

$$\begin{aligned}
u_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \cos\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}\right) \\
&= \cos^2\theta u_{rr} + \cos\theta u_r(0) - \frac{\cos\theta\sin\theta}{r}u_{r\theta} + \frac{\cos\theta\sin\theta}{r^2}u_\theta - \frac{\sin\theta\cos\theta}{r}u_{\theta r} \\
&+ \frac{\sin^2\theta}{r}u_r + \frac{\sin^2\theta}{r^2}u_{\theta\theta} + \frac{\sin\theta\cos\theta}{r^2}u_\theta \\
&= \cos^2\theta u_{rr} - \frac{2\cos\theta\sin\theta}{r}u_{r\theta} + \frac{2\cos\theta\sin\theta}{r^2}u_\theta + \frac{\sin^2\theta}{r}u_r + \frac{\sin^2\theta}{r^2}u_{\theta\theta}
\end{aligned}$$

$$\begin{aligned}
u_{yy} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \sin\theta\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right) + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial u}{\partial r}\sin\theta + \frac{\partial u}{\partial \theta}\frac{\cos\theta}{r}\right) \\
&= \sin^2\theta u_{rr} + \frac{2\cos\theta\sin\theta}{r}u_{r\theta} - \frac{2\cos\theta\sin\theta}{r^2}u_\theta + \frac{\cos^2\theta}{r}u_r + \frac{\cos^2\theta}{r^2}u_{\theta\theta}
\end{aligned}$$

The derivative with respect to z is the same. Hence, adding gives:

$$\begin{aligned}
u_{xx} + u_{yy} + u_{zz} &= \left(\cos^2\theta + \sin^2\theta\right)u_{rr} + \left(\frac{\cos^2\theta + \sin^2\theta}{r}\right)u_r + \left(\frac{\cos^2\theta + \sin^2\theta}{r^2}\right)u_{\theta\theta} + u_{zz} \\
&= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz}
\end{aligned}$$