MATH20411 **Problem sheet 9: Vector Calculus A**

Note: Most of this sheet should be revision.

1. Find the magnitude of the following vector functions:

$$a)\ y\mathbf{i}+x\mathbf{j}, \qquad b)\ x\mathbf{j}, \qquad c)\ \frac{(\mathbf{i}+\mathbf{j})}{\sqrt{2}}, \qquad d)\ \mathbf{i}+y\mathbf{j}.$$

(And sketch them if you are feeling adventurous.)

2. a) Write a formula for a vector function in two dimensions which is in the positive radial direction and whose magnitude is 1.

   b) Write a formula for a vector function in two dimensions whose direction makes an angle of 45^o with the x-axis and whose magnitude at any point is $(x+y)^2$.

   c) Write a formula for a vector function in three dimensions which is in the positive radial direction and whose magnitude is 1.

3. Prove that any given vector function $\mathbf{F}=F_x(x,y,z)\mathbf{i}+F_y(x,y,z)\mathbf{j}+F_z(x,y,z)\mathbf{k}$, satisfies the identity: $\nabla\cdot(\nabla\times\mathbf{F})=0$.

4. Prove that for any scalar function $f(x,y,z)$, $\nabla\times(\nabla f)=0$.

5. Calculate the divergence of each of the following vector fields:

$$a)\ x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}, \quad b)\ yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}, \quad c)\ x\mathbf{i}+y\mathbf{j}+z\mathbf{k}, \quad d)\ \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}}.$$

6. Calculate the curl of each of the following vector fields:

$$a)\ z^2\mathbf{i}+x^2\mathbf{j}-y^2\mathbf{k}, \quad b)\ yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}, \quad c)\ xy\mathbf{i}+y^2\mathbf{j}+yz\mathbf{k}, \quad d)\ \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

7. (Revision) Evaluate the following double integrals on rectangular domains in the x-y plane:

$$a)\int_0^1\int_1^3\left(x^2+y^2\right)\,dydx, \quad b)\int_0^{\pi}\int_1^2 y\sin x\,dydx, \quad c)\int_{-2}^4\int_0^1 xe^y\,dydx.$$

8. (Revision) Let A be the region bounded by the parabolas $y=3x^2$, $y=4-x^2$ and the y-axis. Evaluate the integral:

$$\int\int_A x^2y\,dA.$$

(Hint: Sketch the region A first and calculate suitable limits for the integration.)

9. (Revision) Evaluate the following integrals and sketch the regions in the $x-y$ plane that are determined by the limits of integration:

$$a)\int_0^1\int_0^{x^3} 3\,dydx \qquad b)\int_0^2\int_0^{y^2} y\,dxdy.$$

10. (Revision) For the following double integrals, sketch the region of integration, reverse the order of integration and evaluate both integrals:

$$a)\int_0^1\int_0^x 2-x-y\,dydx, \qquad b)\int_0^3\int_1^{e^x} 2\,dydx, \qquad c)\int_0^{\frac{\pi}{2}}\int_0^{\cos x}\sin x\,dydx.$$

11. Set up and evaluate a volume integral to find the volume of a cuboid described by:

$$V = \{(x, y, z), |a \leq x \leq b,\, c \leq y \leq d,\, e \leq z \leq f\}\,.$$

12. Let

$$V = \{(x, y, z), | -2 \leq x \leq 3,\, 0 \leq y \leq 1,\, 0 \leq z \leq 5\}\,.$$

Evaluate the integral:

$$\int\int\int_V x^2 e^y + xyz\, dV.$$

13. Evaluate the following triple integrals

$$a) \int_{-1}^{1}\int_{0}^{2}\int_{1}^{3} xyz\, dz\, dy\, dx, \quad b) \int_{0}^{1}\int_{0}^{2}\int_{0}^{3} x^2 + y^2 + z^2\, dz\, dy\, dx, \quad c) \int_{-1}^{2}\int_{1}^{z^2}\int_{0}^{y+z} 3yz^2\, dx dy dz.$$

14. Set up and evaluate a volume integral to find the volume of a cylinder of radius a units, with base on the x-y plane and with height b units.

15. Set up and evaluate a volume integral to find the volume of a sphere of radius a units, centered at the origin.