
Before attempting these questions, familiarise yourselves with the explicit and implicit finite difference schemes for the heat equation by reading the handouts. Download the MATLAB codes (as instructed on the handouts) and reproduce the numerical examples.

1. Consider the one-dimensional heat equation:
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq 1, \quad t > 0,$$
subject to the boundary conditions $u(0,t) = 0 = u(1,t)$ and the initial condition $\sin(\pi x) + \sin(3\pi x)$. Using 6 intervals and timestep $k = 0.002$, apply the **explicit** finite difference scheme and calculate an approximation to $u(x,t)$ at the grid points at the first time level $t_1 = 0.002$. Sketch the results and compare your answers with the exact solutions. (Hint: the exact solution can be obtained with separation of variables.)

2. Show that if the exact solution is sufficiently continuous, the truncation error, T_j^m, at a point (x_j, t_m), for the explicit finite difference scheme (consisting of a forward difference in time and centered difference in space) applied to the heat equation is:
$$T_j^m = \frac{k}{2} u_{tt}(x_j, t_m) - \frac{h^2}{12} u_{xxxx}(x_j, t_m) + \text{terms with higher powers of } h \text{ and } k.$$
Hence show that for a fixed value of $\nu = \frac{k}{h^2}$, the scheme is first order accurate with respect to k.

3. Let e_j^m denote the error $u(x_j, t_m) - U_j^m$ and let $E^m = \max_j \mid e_j^m \mid$ denote the maximum error at time level t_m and let $T^m = \max_j \mid T_j^m \mid$ denote the maximum truncation error at time level t_m. Using the expression for the truncation error and the definition of the numerical scheme, show that:
   a) $e_j^{m+1} = (1 - 2\nu) e_j^m + \nu e_{j+1}^m + \nu e_{j-1}^m + k T_j^m, \quad j = 1 : N-1, \ m = 1, 2, \ldots$
   b) if $\nu \leq \frac{1}{2}$, and T^* is an upper-bound for T^i at each time level, show that the maximum error at $t = t_m$ satisfies $E^m \leq t_m T^*$.

4. Consider, again, the one-dimensional heat equation, on the interval $[0, 1]$, subject to the boundary conditions $u(0,t) = 0 = u(1,t)$ and the initial condition $\sin(\pi x)$. Using 6 intervals and the timestep $\Delta t = 0.01s$, apply the **implicit** finite difference scheme from the lectures. Derive the tridiagonal system of equations to be solved to compute the numerical solution at the first time step. In MATLAB, solve this system using `trisolve.m`.